

# Long time asymptotics of non-symmetric random walks on crystal lattices

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## Abstract

In the present paper, we study long time asymptotics of non-symmetric random walks on crystal lattices from a view point of discrete geometric analysis due to Kotani and Sunada [11, 23]. We observe that the Euclidean metric associated with the standard realization of the crystal lattice, called the Albanese metric, naturally appears in the asymptotics. In the former half of the present paper, we establish two kinds of (functional) central limit theorems for random walks. We first show that the Brownian motion on the Euclidean space with the Albanese metric appears as the scaling limit of the usual central limit theorem for the random walk. Next we introduce a family of random walks which interpolates between the original non-symmetric random walk and the symmetrized one. We then capture the Brownian motion with a constant drift of the asymptotic direction on the Euclidean space with the Albanese metric associated with the symmetrized random walk through another kind of central limit theorem for the family of random walks. In the latter half of the present paper, we give a spectral geometric proof of the asymptotic expansion of the  $n$ -step transition probability for the non-symmetric random walk. This asymptotic expansion is a refinement of the local central limit theorem obtained by Sunada [22] and is a generalization of the result in [11] for symmetric random walks on crystal lattices to non-symmetric cases.

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# 1 Introduction

Let  $X = (V, E)$  be an oriented, locally finite connected graph (which may have multiple edges and loops). For an oriented edge  $e \in E$ , the origin and the terminus of  $e$  are denoted by  $o(e)$  and  $t(e)$ , respectively. The inverse edge of  $e \in E$  is denoted by  $\bar{e}$ . Let  $E_x = \{e \in E \mid o(e) = x\}$  be the set of edges with  $o(e) = x \in V$ . A path  $c$  of  $X$  of length  $n$  is a sequence  $c = (e_1, \dots, e_n)$  of oriented edges  $e_i$  with  $o(e_{i+1}) = t(e_i)$  ( $i = 1, \dots, n-1$ ). We denote by  $\Omega_{x,n}(X)$  ( $x \in V, n \in \mathbb{N} \cup \{\infty\}$ ) the set of all paths of length  $n$  for which origin  $o(c) = x$ . For simplicity, we also write  $\Omega_x(X) := \Omega_{x,\infty}(X)$ .

A random walk on  $X$  is a stochastic process with values in  $X$  characterized effectively by a transition probability, a non-negative function  $p : E \rightarrow \mathbb{R}$  satisfying

$$\sum_{e \in E_x} p(e) = 1 \quad (x \in V), \quad p(e) + p(\bar{e}) > 0 \quad (e \in E), \quad (1.1)$$

where  $p(e)$  stands for the probability that a particle at  $o(e)$  moves to  $t(e)$  along the edge  $e$  in one unit time. The transition operator  $L$  on  $X$  associated with the random walk is defined by

$$Lf(x) := \sum_{e \in E_x} p(e)f(t(e)) \quad (x \in V).$$

The  $n$ -step transition probability  $p(n, x, y)$  ( $n \in \mathbb{N}, x, y \in V$ ) is defined by

$$p(n, x, y) := \sum_{c=(e_1, \dots, e_n)} p(e_1) \cdots p(e_n), \quad (1.2)$$

where the sum is taken over all paths  $c = (e_1, \dots, e_n)$  of length  $n$  with the origin  $o(c) = x$  and the terminus  $t(c) = y$ . We mention

$$L^n f(x) = \sum_{y \in V} p(n, x, y)f(y) \quad (x \in V).$$

In a natural manner, the transition probability  $p$  induces the probability measure  $\mathbb{P}_x$  on the set  $\Omega_x(X)$ . The random walk associated with  $p$  is the time homogeneous Markov chain  $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^\infty)$  with values in  $X$  defined by

$$w_n(c) := o(c(n+1)) \quad (n = 0, 1, 2, \dots, c \in \Omega_x(X)),$$

where  $c(n)$  is the  $n$ th edge of the infinite path  $c \in \Omega_x(X)$ . If, in addition, there exists a positive function  $m : V \rightarrow \mathbb{R}$  such that

$$p(e)m(o(e)) = p(\bar{e})m(t(e)) \quad (e \in E),$$

then the random walk is said to be *symmetric* (or *reversible*), and the function  $m$  is called a *reversible measure* for the random walk. Note that  $m$  is uniquely determined up to

a constant multiple. The most canonical symmetric random walk is the simple random walk with the transition probability given by  $p(e) = (\deg o(e))^{-1}$  ( $e \in E$ ).

Studying the long time asymptotics for random walks is a central theme in probability theory. In particular, the central limit theorem (CLT), a generalization of the Laplace–de Moivre theorem, has been studied by many authors in various settings. For basic results, see Spitzer [20], Woess [26], Lawler [17] and literatures therein. As mentioned in Spitzer [20], the periodicity of the graph plays a crucial role to obtain such asymptotics. From this viewpoint, Kotani, Shirai and Sunada [15] applied *discrete geometric analysis* to study the long time asymptotics of symmetric random walks on *crystal lattices*. We also refer to Guivar’ch [2] and Kramli–Szász [16] for related early works. Here  $X = (V, E)$  is called a  $(\Gamma)$ -crystal lattice if there exists an abelian group  $\Gamma$  acting on  $X$  freely and its quotient  $X_0 = (V_0, E_0) = \Gamma \backslash X$  is a finite graph. In other words,  $X$  is an abelian covering graph of a finite graph  $X_0$  for which covering transformation group is  $\Gamma$ . Examples we have in mind are the square lattice  $\mathbb{Z}^d$ , the triangular lattice and the hexagonal lattice (see Figure 1). For simplicity, we assume  $\Gamma$  is torsion free, therefore isomorphic to  $\mathbb{Z}^d$ . If both

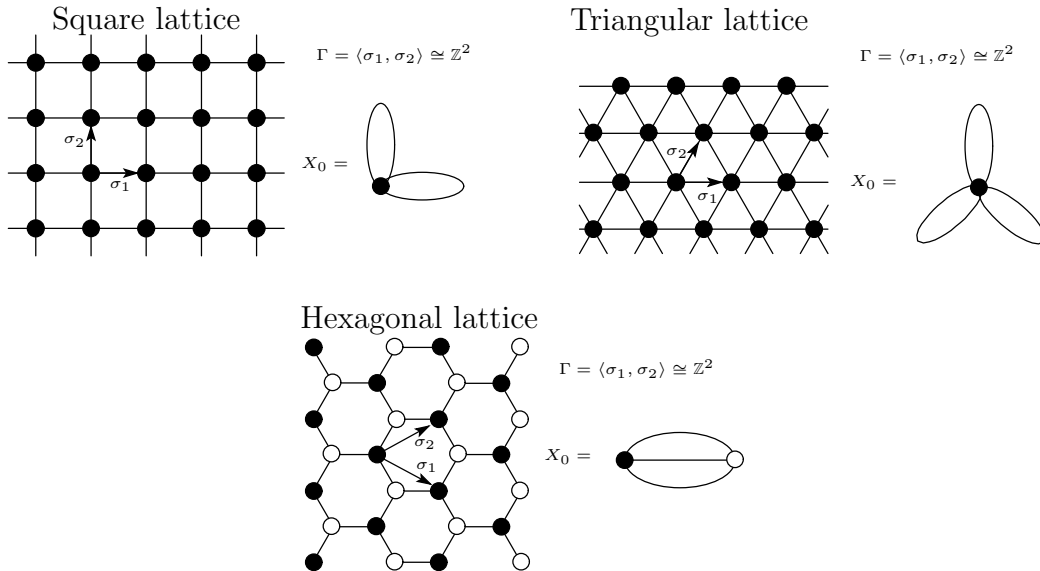


Figure 1: Crystal lattices

$p$  and  $m$  are  $\Gamma$ -invariant, then the symmetric random walk  $\{w_n\}_{n=0}^\infty$  induces a symmetric random walk on the quotient graph  $X_0$  through the covering map  $\pi : X \rightarrow X_0$ , and vice versa.

Later in [11], Kotani and Sunada studied a numerical estimate of the  $n$ -step transition probability  $p(n, x, y)$  for fixed  $x, y \in V$  as  $n \rightarrow \infty$ , called the local central limit theorem (LCLT), for the symmetric random walk on the crystal lattice  $X$  by placing a special emphasis on the geometric feature. Moreover, they also established the asymptotic

expansion

$$p(n, x, y)m(y)^{-1} \sim a_0(X)n^{-d/2} \exp\left(-\frac{d_\Gamma(x, y)^2}{2n}\right) \times (1 + a_1(x, y)n^{-1} + a_2(x, y)n^{-2} + \cdots) \quad \text{as } n \rightarrow \infty, \quad (1.3)$$

where  $d_\Gamma(x, y) := |\Phi_0(y) - \Phi_0(x)|_{\Gamma \otimes \mathbb{R}}$  is a Euclidean pseudo-distance appearing through the *standard realization*  $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$  (cf. [12]). The asymptotic expansion (1.3) yields the CLT

$$\begin{aligned} L^{[nt]}P_{n^{-1/2}}f(x_n) &:= \sum_{y \in V} p([nt], x_n, y) f(n^{-1/2}\Phi_0(y)) \\ &\rightarrow e^{-t\Delta/2}f(\mathbf{x}) \quad (\mathbf{x} \in \Gamma \otimes \mathbb{R}) \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (1.4)$$

where  $\{x_n\}_{n=1}^\infty$  is a sequence in  $V$  such that  $\lim_{n \rightarrow \infty} n^{-1/2}\Phi_0(x_n) = \mathbf{x}$  and  $f$  is a continuous function on  $\Gamma \otimes \mathbb{R}$  vanishing at infinity. Here  $e^{-t\Delta}$  is the heat semigroup generated by the (positive) *Albanese Laplacian*  $\Delta$  on  $\Gamma \otimes \mathbb{R}$ . We mention that (1.4) is also obtained as a special case of [3, 9] by applying Trotter's approximation theory [24]. In terms of probability theory, (1.4) means that, as  $n \rightarrow \infty$ , a sequence of  $\Gamma \otimes \mathbb{R}$ -valued random variables  $\{n^{-1/2}\Phi_0(w_{[nt]})\}_{n=1}^\infty$  ( $t \geq 0$ ) converges to  $B_t$  as  $n \rightarrow \infty$  in law, where  $(B_t)_{t \geq 0}$  is the standard Brownian motion on  $\Gamma \otimes \mathbb{R}$  starting from  $\mathbf{x}$ . Nevertheless, this kind of long time asymptotics for random walks on crystal lattices which are not necessarily symmetric has not been studied satisfactorily although a large deviation principle (LDP) is obtained in [10, 14].

The main purpose of the present paper is to discuss the long time asymptotics for non-symmetric random walks on crystal lattices. In particular, we establish two kinds of (functional) CLTs (Theorems 2.1–2.4) and extend the asymptotic expansion formula (1.3) to non-symmetric cases (Theorem 2.5). These CLTs are extensions of (1.4) to non-symmetric cases and Theorem 2.5 is a refinement of the LCLT for non-symmetric random walks on crystal lattices presented by Sunada [22].

The rest of the present paper is organized as follows: In Section 2, we formulate our problem briefly and state the main results. In Section 3, we make a preparation from the discrete geometric analysis, some ergodic theorems and Trotter's approximation theory. In Section 4, we prove the CLT of the first kind (Theorems 2.1 and 2.2). In Section 5, we prove the CLT of the second kind (Theorems 2.3 and 2.4). In Section 6, we concentrate on giving a spectral geometric proof of Theorem 2.5. In the proof, perturbation arguments on eigenvalues and eigenfunctions of the twisted transition operators  $L_\chi$  and its transposed operator  ${}^tL_\chi$  play crucial roles, and we also make careful use of the classical Laplace method to get the desired asymptotic expansion formula. Finally, in Section 7, we present several concrete examples of the modified standard realization of crystal lattices associated with non-symmetric random walks.

Throughout the present paper,  $C$  denotes a positive constant that may change at every occurrence, and  $O(\cdot)$  stands for the Landau symbol. When the dependence of the  $O(\cdot)$  term is significant, we denote it for example as  $O_N(\cdot)$ .

## 2 Statement of the main results

In this section, we state the main results. (For more details on the discrete geometric analysis, see Section 3.) Let  $p : E \rightarrow \mathbb{R}$  be a  $\Gamma$ -invariant non-negative function satisfying (1.1). The random walk on the crystal lattice  $X$  associated with the transition probability  $p$  induces a random walk on the quotient graph  $X_0$  through the covering map  $\pi : X \rightarrow X_0$ , and the  $n$ -step transition probability  $p(n, x, y)$  ( $n \in \mathbb{N}$ ,  $x, y \in V_0$ ) is also defined by (1.2). Throughout the present paper, we assume that the random walk on  $X_0$  is *irreducible*, that is, for every  $x, y \in V_0$ , there exists some  $n = n(x, y) \in \mathbb{N}$  such that  $p(n, x, y) > 0$ . Then by applying Perron–Frobenius theorem (cf. Parry–Pollicott [18, Theorem 2.2]), we find a unique positive function  $m : V_0 \rightarrow \mathbb{R}$ , called the *invariant probability measure*, satisfying

$$m(x) = \sum_{e \in (E_0)_x} p(\bar{e})m(t(e)) \quad (x \in V_0), \quad \sum_{x \in V_0} m(x) = 1. \quad (2.1)$$

We also write  $m : V \rightarrow \mathbb{R}$  for the ( $\Gamma$ -invariant) lift of the invariant measure  $m$ .

For a topological space  $\mathcal{T}$ , let  $C_\infty(\mathcal{T})$  denote the space of continuous functions on  $\mathcal{T}$  vanishing at infinity. This space is endowed with usual uniform topology  $\|\cdot\|_\infty$ . We note that a graph has the discrete topology defined by the graph distance. Let  $H_1(X_0, \mathbb{R})$  and  $H^1(X_0, \mathbb{R})$  be the first homology group and the first cohomology group on  $X_0$ , respectively. Let  $\rho_{\mathbb{R}}$  be the canonical surjective linear map from  $H_1(X_0, \mathbb{R})$  to  $\Gamma \otimes \mathbb{R}$ . We define the *homological direction*  $\gamma_p$  by

$$\gamma_p := \sum_{e \in E_0} p(e)m(o(e))e \in H_1(X_0, \mathbb{R}),$$

and call  $\rho_{\mathbb{R}}(\gamma_p) \in \Gamma \otimes \mathbb{R}$  the *asymptotic direction*. It should be noted that  $\gamma_p = 0$  if and only if the random walk on  $X_0$  is symmetric, and  $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$  does not always imply the symmetry of the random walk on  $X$ . We also introduce the *modified harmonic realization*  $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$  by

$$\sum_{e \in E_x} p(e)(\Phi_0(t(e)) - \Phi_0(o(e))) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V). \quad (2.2)$$

Note that  $\Phi_0$  is uniquely determined up to translation (cf. [14, page 854]).

To present the CLT of the first kind, we need to introduce the *transition-shift operator*  $\mathcal{L}_{\gamma_p}$  acting on  $C_\infty(X \times H_1(X_0, \mathbb{R}))$  by

$$\mathcal{L}_{\gamma_p} f(x, \mathbf{z}) := \sum_{e \in E_x} p(e)f(t(e), \mathbf{z} + \gamma_p) \quad (x \in V, \mathbf{z} \in H_1(X_0, \mathbb{R})),$$

and the *approximation operator*  $\mathcal{P}_\varepsilon : C_\infty(\Gamma \otimes \mathbb{R}) \rightarrow C_\infty(X \times H_1(X_0, \mathbb{R}))$  ( $0 \leq \varepsilon \leq 1$ ) by

$$\mathcal{P}_\varepsilon f(x, \mathbf{z}) := f(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))).$$

Then the CLT of the first kind is stated as follows:

**Theorem 2.1** *For  $0 \leq s < t$  and  $f \in C_\infty(\Gamma \otimes \mathbb{R})$ ,*

$$\lim_{n \rightarrow \infty} \left\| \mathcal{L}_{\gamma_p}^{[nt] - [ns]} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{-\frac{t-s}{2} \Delta} f \right\|_\infty = 0, \quad (2.3)$$

where  $\Delta$  is the (positive) Laplacian associated with the Albanese metric  $g_0$  on  $\Gamma \otimes \mathbb{R}$ . (See Section 3 for the definition of the Albanese metric  $g_0$ .)

In particular, for any sequence  $\{(x_n, \mathbf{z}_n)\}_{n=1}^\infty$  in  $V \times H_1(X_0, \mathbb{R})$  with

$$\lim_{n \rightarrow \infty} n^{-1/2} (\Phi_0(x_n) - \rho_{\mathbb{R}}(\mathbf{z}_n)) = \mathbf{x} \in \Gamma \otimes \mathbb{R},$$

and for any  $f \in C_\infty(\Gamma \otimes \mathbb{R})$ , we have

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, \mathbf{z}_n) = e^{-t\Delta/2} f(\mathbf{x}) := \int_{\Gamma \otimes \mathbb{R}} G_t(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \quad (t > 0), \quad (2.4)$$

where

$$G_t(\mathbf{x}) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\mathbf{x}|_{g_0}^2}{2t}\right) \quad (\mathbf{x} \in \Gamma \otimes \mathbb{R})$$

is the fundamental solution of the heat equation

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = -\frac{1}{2} \Delta u(t, \mathbf{x}).$$

We mention that this theorem still holds for the approximation operator  $\mathcal{P}_\varepsilon$  given by a general periodic realization  $\Phi : X \rightarrow \Gamma \otimes \mathbb{R}$  (cf. Ishiwata [3]).

Now we set a reference point  $x_* \in V$  such that  $\Phi_0(x_*) = \mathbf{0}$ , and put

$$\xi_n(c) := \Phi_0(w_n(c)) \quad (n = 0, 1, 2, \dots, c \in \Omega_{x_*}(X)).$$

We then obtain a  $\Gamma \otimes \mathbb{R}$ -valued random walk  $(\Omega_{x_*}(X), \mathbb{P}_{x_*}, \{\xi_n\}_{n=0}^\infty)$ . Let  $\mathcal{X}_t^{(n)}$  ( $t \geq 0, n \in \mathbb{N}$ ) be a map from  $\Omega_{x_*}(X)$  to  $\Gamma \otimes \mathbb{R}$  given by

$$\mathcal{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}} (\xi_{[nt]}(c) - [nt] \rho_{\mathbb{R}}(\gamma_p)) \quad (c \in \Omega_{x_*}(X)).$$

Then (2.4) can be rewritten as

$$\lim_{n \rightarrow \infty} \sum_{c \in \Omega_{x_*}(X)} f(\mathcal{X}_t^{(n)}(c)) \mathbb{P}_{x_*}(\{c\}) = \int_{\mathbf{W}} f(\mathbf{w}_t) \mathbf{P}^W(d\mathbf{w}),$$

where  $\mathbf{P}^W$  is the *Wiener measure* on  $\mathbf{W} := C_0([0, \infty), \Gamma \otimes \mathbb{R})$ . As the piecewise linear interpolation of  $(\mathcal{X}_t^{(n)})_{t \geq 0}$ , we define a map  $\mathbf{X}^{(n)} : \Omega_{x_*}(X) \rightarrow \mathbf{W}$  by

$$\mathbf{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}} \left\{ \xi_{[nt]}(c) + (nt - [nt]) (\xi_{[nt]+1}(c) - \xi_{[nt]}(c)) - nt \rho_{\mathbb{R}}(\gamma_p) \right\} \quad (t \geq 0).$$

Let  $\mathbf{P}^{(n)}$  be the probability measure on  $(\mathbf{W}, \mathcal{B}(\mathbf{W}))$  induced by  $\mathbf{X}^{(n)}$ .

The following theorem is the functional CLT (i.e., Donsker's invariance principle) for the random walk  $\{\xi_n\}_{n=0}^\infty$ .

**Theorem 2.2** *The sequence  $\{\mathbf{P}^{(n)}\}_{n=1}^\infty$  converges weakly to the Wiener measure  $\mathbf{P}^W$  as  $n \rightarrow \infty$ . Namely,  $\mathbf{X}^{(n)}$  converges to a  $\Gamma \otimes \mathbb{R}$ -valued standard Brownian motion  $(B_t)_{t \geq 0}$  with  $B_0 = \mathbf{0}$  in law.*

We next present another kind of CLT for a family of random walks. (See Durrett [1] and Trotter [24] for related results.) For the transition probability  $p$ , we introduce a family of  $(\Gamma$ -invariant) transition probabilities  $\{p_\varepsilon\}_{0 \leq \varepsilon \leq 1}$  on  $X$  by

$$p_\varepsilon(e) := p_0(e) + \varepsilon q(e) \quad (e \in E), \quad (2.5)$$

where

$$p_0(e) := \frac{1}{2} \left( p(e) + \frac{m(t(e))}{m(o(e))} p(\bar{e}) \right), \quad q(e) := \frac{1}{2} \left( p(e) - \frac{m(t(e))}{m(o(e))} p(\bar{e}) \right).$$

We note that  $\{p_\varepsilon\}_{0 \leq \varepsilon \leq 1}$  is the interpolation between the original transition probability  $p = p_1$  and the  $m$ -symmetric probability  $p_0$  such that the homological direction  $\gamma_{p_\varepsilon}$  equals  $\varepsilon \gamma_p$  for every  $0 \leq \varepsilon \leq 1$  (see Lemma 5.1 below for details).

We denote by  $L_{(\varepsilon)}$  the transition operator associated with  $p_\varepsilon$ . We denote by  $g_0^{(\varepsilon)}$  the corresponding Albanese metric on  $\Gamma \otimes \mathbb{R}$ . If we need to emphasize the flat metric  $g_0^{(\varepsilon)}$ , we write  $(\Gamma \otimes \mathbb{R})_{(\varepsilon)}$  for  $(\Gamma \otimes \mathbb{R}, g_0^{(\varepsilon)})$ . We define another approximation operator  $P_\varepsilon : C_\infty((\Gamma \otimes \mathbb{R})_{(0)}) \rightarrow C_\infty(X)$  by

$$P_\varepsilon f(x) := f(\varepsilon \Phi_0^{(\varepsilon)}(x)) \quad (x \in V),$$

where  $\Phi_0^{(\varepsilon)} : X \rightarrow \Gamma \otimes \mathbb{R}$  is the modified harmonic realization associated with  $p_\varepsilon$ .

Then the CLT of the second kind is stated as follows:

**Theorem 2.3** *For  $0 \leq s < t$  and  $f \in C_\infty((\Gamma \otimes \mathbb{R})_{(0)})$ ,*

$$\lim_{n \rightarrow \infty} \left\| L_{(n^{-1/2})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} e^{-(t-s)(\frac{1}{2}\Delta_{(0)} - \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} \rangle_{g_0^{(0)}})} f \right\|_\infty = 0,$$

where  $\Delta_{(0)}$  and  $\nabla_{(0)}$  stand for the (positive) Laplacian and the gradient on  $(\Gamma \otimes \mathbb{R})_{(0)}$ , respectively. In particular, for any sequence  $\{x_n\}_{n=1}^\infty$  in  $V$  with

$$\lim_{n \rightarrow \infty} n^{-1/2} \Phi_0^{(n^{-1/2})}(x_n) = \mathbf{x} \in (\Gamma \otimes \mathbb{R})_{(0)},$$

and for any  $f \in C_\infty((\Gamma \otimes \mathbb{R})_{(0)})$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{y \in V} p_{n^{-1/2}}([nt], x_n, y) f(n^{-1/2} \Phi_0^{(n^{-1/2})}(y)) \\ &= e^{-t(\frac{1}{2} \Delta_{(0)} - \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} \rangle_{g_0^{(0)}})} f(\mathbf{x}) = \int_{(\Gamma \otimes \mathbb{R})_{(0)}} H_t(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \end{aligned} \quad (2.6)$$

where

$$H_t(\mathbf{x}) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\mathbf{x} - \rho_{\mathbb{R}}(\gamma_p)t|_{g_0^{(0)}}^2}{2t}\right)$$

is the fundamental solution of the heat equation

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = -\frac{1}{2} \Delta_{(0)} u(t, \mathbf{x}) + \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} u(t, \mathbf{x}) \rangle_{g_0^{(0)}}.$$

Just like Theorems 2.1 and 2.2, this theorem also implies a functional CLT. Set a reference point  $x_* \in V$  such that  $\Phi_0^{(\varepsilon)}(x_*) = \mathbf{0}$  for all  $0 \leq \varepsilon \leq 1$ . Putting

$$\xi_n^{(\varepsilon)}(c) := \Phi_0^{(\varepsilon)}(w_n(c)) \quad (0 \leq \varepsilon \leq 1, n = 0, 1, 2, \dots, c \in \Omega_{x_*}(X)),$$

we also obtain a  $\Gamma \otimes \mathbb{R}$ -valued random walk  $(\Omega_{x_*}(X), \mathbb{P}_{x_*}^{(\varepsilon)}, \{\xi_n^{(\varepsilon)}\}_{n=0}^\infty)$  associated with  $p_\varepsilon$ . Let  $\mathcal{Y}_t^{(\varepsilon, n)}$  ( $t \geq 0, n \in \mathbb{N}, 0 \leq \varepsilon \leq 1$ ) be a map from  $\Omega_{x_*}(X)$  to  $\Gamma \otimes \mathbb{R}$  given by

$$\mathcal{Y}_t^{(\varepsilon, n)}(c) := \frac{1}{\sqrt{n}} \xi_{[nt]}^{(\varepsilon)}(c) \quad (c \in \Omega_{x_*}(X)).$$

Then (2.6) can be rewritten as

$$\lim_{n \rightarrow \infty} \sum_{c \in \Omega_{x_*}(X)} f(\mathcal{Y}_t^{(n^{-1/2}, n)}(c)) \mathbb{P}_{x_*}(\{c\}) = \int_{\mathbf{W}_{(0)}} f(\mathbf{w}_t) \mathbf{Q}(d\mathbf{w}),$$

where  $\mathbf{Q}$  is the probability measure on  $\mathbf{W}_{(0)} := C_0([0, \infty), (\Gamma \otimes \mathbb{R})_{(0)})$  induced by  $(B_t + \rho_{\mathbb{R}}(\gamma_p)t)_{t \geq 0}$ . Here  $(B_t)_{t \geq 0}$  is a  $(\Gamma \otimes \mathbb{R})_{(0)}$ -valued standard Brownian motion with  $B_0 = \mathbf{0}$ . We define a map  $\mathbf{Y}^{(\varepsilon, n)} : \Omega_{x_*}(X) \rightarrow \mathbf{W}_{(0)}$  by

$$\mathbf{Y}_t^{(\varepsilon, n)}(c) := \frac{1}{\sqrt{n}} \left\{ \xi_{[nt]}^{(\varepsilon)}(c) + (nt - [nt]) (\xi_{[nt]+1}^{(\varepsilon)}(c) - \xi_{[nt]}^{(\varepsilon)}(c)) \right\} \quad (t \geq 0, c \in \Omega_{x_*}(X)),$$

which is the piecewise linear interpolation of  $(\mathcal{Y}_t^{(\varepsilon, n)})_{t \geq 0}$ . Let  $\mathbf{Q}^{(\varepsilon, n)}$  be the probability measure on  $(\mathbf{W}_{(0)}, \mathcal{B}(\mathbf{W}_{(0)}))$  induced by  $\mathbf{Y}^{(\varepsilon, n)}$ .

The following theorem is a functional CLT for the family of random walks  $\{\xi_n^{(\varepsilon)}\}_{n=0}^\infty$  ( $0 \leq \varepsilon \leq 1$ ).

**Theorem 2.4** *The sequence  $\{\mathbf{Q}^{(n^{-1/2}, n)}\}_{n=1}^\infty$  converges weakly to  $\mathbf{Q}$  as  $n \rightarrow \infty$ . Namely,  $(\mathbf{Y}_t^{(n^{-1/2}, n)})_{t \geq 0}$  converges to a  $(\Gamma \otimes \mathbb{R})_{(0)}$ -valued standard Brownian motion with drift  $\rho_{\mathbb{R}}(\gamma_p)$  starting from  $\mathbf{0}$  as  $n \rightarrow \infty$  in law.*



Next, we discuss the precise asymptotic behavior of the  $n$ -step transition probability  $p(n, x, y)$  ( $x, y \in V$ ) of the non-symmetric random walk  $\{w_n\}_{n=0}^\infty$  on  $X$  as  $n \rightarrow \infty$ . Here we impose the condition that the random walk  $\{w_n\}_{n=0}^\infty$  on  $X$  is *irreducible*. Note that this condition implies the *irreducibility on  $X_0$*  imposed above. Conversely, the irreducibility of the random walk on  $X_0$  does not imply the irreducibility of the random walk on  $X$ . We define the *period* of the random walk  $\{w_n\}_{n=0}^\infty$  by  $K := \gcd\{n \in \mathbb{N} \mid p(n, x, x) > 0\}$ . It should be noted that  $K$  is independent of  $x \in V$  by the irreducibility condition. Sunada [22, page 121] presented the LCLT

$$(2\pi n)^{d/2} p(n, x, y) m(y)^{-1} \sim K \operatorname{vol}(\operatorname{Alb}^\Gamma) \exp \left( - \frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right) \quad (x, y \in V) \quad (2.7)$$

as  $n \rightarrow \infty$ , where  $\operatorname{vol}(\operatorname{Alb}^\Gamma)$  stands for the volume of the  $\Gamma$ -Albanese torus  $(\Gamma \otimes \mathbb{R} / \Gamma \otimes \mathbb{Z}, g_0)$ .

As a refinement of (2.7), we obtain the following precise asymptotics of  $p(n, x, y)$ .

**Theorem 2.5** *Suppose that the random walk  $\{w_n\}_{n=0}^\infty$  on  $X$  is irreducible with period  $K$ . Let  $V = \coprod_{k=0}^{K-1} A_k$  be the corresponding  $K$ -partition of  $V$ . Then for any  $x \in A_i$  and  $y \in A_j$  ( $0 \leq i, j \leq K-1$ ), we have*

$$p(n, x, y) = 0 \quad (n \neq Kl + j - i)$$

and

$$(2\pi n)^{d/2} p(n, x, y) m(y)^{-1} = K \operatorname{vol}(\operatorname{Alb}^\Gamma) \exp \left( - \frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right) \times \left( 1 + a_1(\pi(x), \pi(y), \gamma_p; \Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)) n^{-1} \right) + O(n^{-3/2}) \quad (2.8)$$

as  $n = Kl + j - i \rightarrow \infty$  uniformly for  $x, y$  with  $|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \leq Cn^{1/6}$ .

The coefficient  $a_1 = a_1(\pi(x), \pi(y), \gamma_p; \Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p))$  is nothing but the one of the term  $n^{-1}$  in the power series expansion of

$$U_n := \frac{(2\pi n)^{d/2} p(n, x, y) m(y)^{-1}}{K \operatorname{vol}(\operatorname{Alb}^\Gamma) \exp \left( - \frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right)}$$

in  $n^{-1}$ , and its explicit expression will be given in Theorem 6.8 below. Using the same argument as in Section 6, we can also give the coefficient  $a_j$  of the term  $n^{-j}$  for any  $j \geq 2$  in the power series expansion of  $U_n$ . We do not write the explicit asymptotic expansion of  $p(n, x, y)$  in the present paper, because of its complication.

We should also mention that Uchiyama and his coauthor [7, 25] recently obtained this kind of asymptotic expansion formula for non-symmetric random walks on periodic graphs (i.e., crystal lattices) realized in Euclidean space under the *zero mean condition*. Their approach is purely probabilistic, and roughly speaking, the zero mean condition is translated into  $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$  in our geometric framework. Thus Theorem 2.5 can be regarded as a generalization of their asymptotic expansion including the case  $\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}$ .

### 3 Preliminaries

#### 3.1 Discrete geometric analysis on graphs

In this subsection, we review some basic facts of *discrete geometric analysis* on graphs. We refer to Kotani–Sunada [13], Sunada [21, 23] and references therein for details.

First of all, we consider an irreducible random walk on  $X_0 = (V_0, E_0)$  with the transition probability  $p : E_0 \rightarrow [0, 1]$ . By the Perron–Frobenius theorem, there exists a unique positive function  $m : V_0 \rightarrow \mathbb{R}$ , called the invariant probability measure, satisfying (2.1). We put

$$\tilde{m}(e) = m(o(e))p(e) \quad (e \in E_0).$$

It is easy to see that  $\tilde{m} : E_0 \rightarrow \mathbb{R}$  satisfies  $\sum_{e \in E_0} \tilde{m}(e) = 1$  and

$$m(x) = \sum_{e \in (E_0)_x} \tilde{m}(e) = \sum_{e \in (E_0)_x} \tilde{m}(\bar{e}) \quad (x \in V_0). \quad (3.1)$$

When  $\tilde{m}(e) = \tilde{m}(\bar{e})$ , the random walk is said to be *(m-)symmetric*.

We consider the 0-chain group

$$C_0(X_0, \mathbb{R}) = \left\{ \sum_{x \in V_0} a_x x \mid a_x \in \mathbb{R} \right\}$$

and the 1-chain group

$$C_1(X_0, \mathbb{R}) = \left\{ \sum_{e \in E_0} a_e e \mid a_e \in \mathbb{R} \right\},$$

where the relation  $\bar{e} = -e$  is imposed for  $e \in E_0$ . The boundary operator  $\partial : C_1(X_0, \mathbb{R}) \rightarrow C_0(X_0, \mathbb{R})$  is defined by  $\partial(e) := t(e) - o(e)$ . The first homology group  $H_1(X_0, \mathbb{R})$  is the kernel  $\text{Ker}(\partial) \subset C_1(X_0, \mathbb{R})$ .  $H_1(X_0, \mathbb{Z})$  is also defined by replacing  $\mathbb{R}$  by  $\mathbb{Z}$ .

We define the 0-cochain group

$$C^0(X_0, \mathbb{R}) := \{f : V_0 \rightarrow \mathbb{R}\}$$

with the inner product

$$\langle f_1, f_2 \rangle_0 = \sum_{x \in V_0} f_1(x) f_2(x) \quad (f_1, f_2 \in C^0(X_0, \mathbb{R})),$$

and the 1-cochain group

$$C^1(X_0, \mathbb{R}) := \{\omega : E_0 \rightarrow \mathbb{R} \mid \omega(\bar{e}) = -\omega(e)\}$$

with the inner product

$$\langle \omega_1, \omega_2 \rangle_1 = \frac{1}{2} \sum_{e \in E_0} \omega_1(e) \omega_2(e) \quad (\omega_1, \omega_2 \in C^1(X_0, \mathbb{R})).$$

A 1-cochain is occasionally called a 1-form on  $X_0$ .

We define the difference operator  $d : C^0(X_0, \mathbb{R}) \rightarrow C^1(X_0, \mathbb{R})$  by

$$df(e) := f(t(e)) - f(o(e)) \quad (e \in E_0),$$

and the first cohomology group  $H^1(X_0, \mathbb{R}) := C^1(X_0, \mathbb{R})/\text{Im}(d)$ . Note that  $H^1(X_0, \mathbb{R})$  is the dual of the first homology group  $H_1(X_0, \mathbb{R})$ . We also define the transition operator  $L : C^0(X_0, \mathbb{R}) \rightarrow C^0(X_0, \mathbb{R})$  by

$$Lf(x) := (I - \delta_p d)f(x) = \sum_{e \in (E_0)_x} p(e)f(t(e)) \quad (x \in V_0),$$

where  $\delta_p : C^1(X_0, \mathbb{R}) \rightarrow C^0(X_0, \mathbb{R})$  is given by

$$(\delta_p \omega)(x) := - \sum_{e \in (E_0)_x} p(e)\omega(e) \quad (x \in V_0).$$

We easily see

$$p(n, x, y) = L^n \delta_y(x) \quad (n \in \mathbb{N}, x, y \in V_0),$$

where  $\delta_y(\cdot)$  is the Dirac delta function with pole at  $y$ .

Let  ${}^tL : C^0(X_0, \mathbb{R}) \rightarrow C^0(X_0, \mathbb{R})$  be the transposed operator of the transition operator  $L$  with respect to the inner product  $\langle \cdot, \cdot \rangle_0$ . The explicit form is given by

$${}^tLf(x) := \sum_{e \in (E_0)_x} p(\bar{e})f(t(e)) \quad (x \in V_0).$$

We note that (2.1) is written as

$${}^tLm(x) = m(x), \quad \sum_{x \in V_0} m(x) = 1.$$

It means that the invariant measure  $m$  is an eigenfunction of  ${}^tL$  for the maximal positive eigenvalue 1.

Now we provide the notion of *modified harmonic 1-forms* on  $X_0$  associated with the transition probability  $p$ . We introduce the random variables  $\eta_i : \Omega_x(X_0) \rightarrow C_1(X_0, \mathbb{R})$  ( $i \in \mathbb{N}, x \in V_0$ ) defined by  $\eta_i(c) := e_i$  for  $c = (e_1, e_2, \dots) \in \Omega_x(X_0)$ , and set

$$\gamma_p := \sum_{e \in E_0} \tilde{m}(e)e \in C_1(X_0, \mathbb{R}).$$

A simple application of the ergodic theorem leads to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \eta_i(c) = \gamma_p, \quad \mathbb{P}_x\text{-a.e. } c \in \Omega_x(X_0).$$

By virtue of (3.1), we observe that  $\partial\gamma_p = 0$  and hence  $\gamma_p \in H_1(X_0, \mathbb{R})$ . It provides a quantity to measure a homological drift of the random walk and we call it the *homological direction* of the given random walk. Straightforwardly,  $\gamma_p = 0$  if and only if  $p$  gives a symmetric random walk, i.e.,  $\tilde{m}(e) = \tilde{m}(\bar{e})$ . A 1-form  $\omega$  is said to be *modified harmonic* if

$$\delta_p \omega(x) + \langle \gamma_p, \omega \rangle = 0 \quad (x \in V_0), \quad (3.2)$$

where it should be noted that  $\langle \gamma_p, \omega \rangle := {}_{C^1(X_0, \mathbb{R})} \langle \gamma_p, \omega \rangle_{C^1(X_0, \mathbb{R})}$  is constant as a function on  $V_0$ . We denote by  $\mathcal{H}^1(X_0)$  the set of modified harmonic 1-forms, and equip  $\mathcal{H}^1(X_0)$  with the inner product

$$\langle\langle \omega_1, \omega_2 \rangle\rangle := \sum_{e \in E_0} \omega_1(e) \omega_2(e) \tilde{m}(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle \quad (\omega_1, \omega_2 \in \mathcal{H}^1(X_0)). \quad (3.3)$$

Then the corresponding norm  $\|\cdot\|$  is given by

$$\|\omega\|^2 := \langle\langle \omega, \omega \rangle\rangle = \sum_{e \in E_0} |\omega(e)|^2 \tilde{m}(e) - \langle \gamma_p, \omega \rangle^2 \quad (\omega \in \mathcal{H}^1(X_0)).$$

By the discrete Hodge–Kodaira theorem (cf. [14, Lemma 5.2]), we may identify  $H^1(X_0, \mathbb{R})$  and  $H^1(X_0, \mathbb{Z})$  with  $\mathcal{H}^1(X_0)$  and

$$\left\{ \omega \in \mathcal{H}^1(X_0) \mid \int_c \omega := \sum_{i=1}^n \omega(e_i) \in \mathbb{Z} \text{ for every closed path } c = (e_1, \dots, e_n) \text{ in } X_0 \right\},$$

respectively. Using this identification, we obtain an inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $H^1(X_0, \mathbb{R})$ .

**Remark 3.1** In [11], Kotani and Sunada considered a symmetric random walk (i.e.,  $\gamma_p = 0$ ) and defined the Albanese metric on  $\Gamma \otimes \mathbb{R}$  associated with the inner product

$$\frac{1}{2} \sum_{e \in E_0} \omega_1(e) \omega_2(e) \tilde{m}(e) \quad (\omega_1, \omega_2 \in \mathcal{H}^1(X_0)).$$

Due to this difference of the definition of the Albanese metric, the Laplacian  $\Delta$  on  $\Gamma \otimes \mathbb{R}$  appears in long time asymptotics instead of  $\Delta/2$ .

Let  $X = (V, E)$  be a  $(\Gamma)$ -crystal lattice, that is,  $X$  is an abelian covering graph of a finite graph  $X_0$  in which covering transformation group is  $\Gamma$ . Let  $p : E \rightarrow \mathbb{R}$  and  $m : V \rightarrow \mathbb{R}$  be the lift of the transition probability  $p : E_0 \rightarrow \mathbb{R}$  and the invariant measure  $m : V_0 \rightarrow \mathbb{R}$ , respectively. Namely,

$$p(\sigma e) = p(e), \quad m(\sigma x) = m(x) \quad (e \in E, x \in V, \sigma \in \Gamma).$$

We denote by  $\pi : X \rightarrow X_0$  the covering map, and by  $\rho : H_1(X_0, \mathbb{Z}) \rightarrow \Gamma$  the surjective homomorphism associated with the covering map  $\pi$ . We extend  $\rho$  to the surjective linear

map  $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R}$ . Then we may consider the injective linear map  ${}^t\rho_{\mathbb{R}} : \text{Hom}(\Gamma, \mathbb{R}) \rightarrow H^1(X_0, \mathbb{R})$  by

$${}^t\rho_{\mathbb{R}} : \omega \in \text{Hom}(\Gamma, \mathbb{R}) \mapsto {}^t\rho_{\mathbb{R}}(\omega)(\cdot) := \omega(\rho_{\mathbb{R}}(\cdot)) \in H^1(X_0, \mathbb{R}),$$

where  $\text{Hom}(\Gamma, \mathbb{R})$  denotes the linear space of homomorphisms of  $\Gamma$  into  $\mathbb{R}$ . Using the maps  ${}^t\rho_{\mathbb{R}}$  and  $\rho_{\mathbb{R}}$ , we identify  $\text{Hom}(\Gamma, \mathbb{R})$  with the subspace  $\text{Image}({}^t\rho_{\mathbb{R}})$  in  $H^1(X_0, \mathbb{R})$  and  $\Gamma \otimes \mathbb{R}$  with the quotient linear subspace of  $H_1(X_0, \mathbb{R})$ . Throughout the present paper, we shall denote  ${}^t\rho_{\mathbb{R}}(\omega) \in H^1(X_0, \mathbb{R})$  by the same symbol  $\omega$  for brevity. We restrict the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $H^1(X_0, \mathbb{R})$  to the subspace  $\text{Hom}(\Gamma, \mathbb{R})$ , and then take up the dual inner product  $\langle \cdot, \cdot \rangle_{\text{alb}}$  on  $\Gamma \otimes \mathbb{R}$ . The flat metric on  $\Gamma \otimes \mathbb{R}$  induced from this inner product is called the *Albanese metric* and is denoted by  $g_0$ . This procedure is summarized in the following diagram:

$$\begin{array}{ccc} (\Gamma \otimes \mathbb{R}, g_0) & \xleftarrow{\rho_{\mathbb{R}}} & H_1(X_0, \mathbb{R}) \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \text{Hom}(\Gamma, \mathbb{R}) & \xrightarrow{{}^t\rho_{\mathbb{R}}} & H^1(X_0, \mathbb{R}) \cong (\mathcal{H}^1(X_0), \langle\langle \cdot, \cdot \rangle\rangle) \end{array}$$

We write  $\text{Alb}^{\Gamma}$  for  $(\Gamma \otimes \mathbb{R} / \Gamma \otimes \mathbb{Z}, g_0)$ , and call it the  $\Gamma$ -*Albanese torus* associated with  $(X, \Gamma)$ .

Now we realize  $X$  in  $\Gamma \otimes \mathbb{R}$  equipped with the Albanese metric  $g_0$  in a standard way. A (piecewise linear) map  $\Phi : X \rightarrow \Gamma \otimes \mathbb{R}$  is said to be a *periodic realization* of  $X$  if it satisfies

$$\Phi(\sigma x) = \Phi(x) + \sigma \otimes 1 \quad (x \in X, \sigma \in \Gamma).$$

We may define a special periodic realization  $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$  by  $\Phi_0(x_*) = \mathbf{0}$  for a fixed base point  $x_* \in V$  and

$$\langle \omega, \Phi_0(x) \rangle_{\Gamma \otimes \mathbb{R}} = \int_{x_*}^x \tilde{\omega} \quad (\omega \in \text{Hom}(\Gamma, \mathbb{R})), \quad (3.4)$$

where  $\tilde{\omega}$  is the lift of  $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in H^1(X_0, \mathbb{R})$  to  $X$ . Here

$$\int_{x_*}^x \tilde{\omega} = \int_c \tilde{\omega} := \sum_{i=1}^n \tilde{\omega}(e_i)$$

for a path  $c = (e_1, \dots, e_n)$  with  $o(e_1) = x_*$  and  $t(e_n) = x$ . It should be noted that this line integral does not depend on the choice of a path  $c$ .

One of the special properties of  $\Phi_0$  is that it is a vector-valued *modified-harmonic function* on  $X$  in the sense that

$$L\Phi_0(x) - \Phi_0(x) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V). \quad (3.5)$$

Indeed, for every  $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in H^1(X_0, \mathbb{R})$ , the modified harmonicity (3.2),  $\Gamma$ -invariance of the transition probability  $p$  and the identity (3.4) imply

$$\begin{aligned}
{}_{\text{Hom}(\Gamma, \mathbb{R})} \langle \omega, L\Phi_0(x) - \Phi_0(x) \rangle_{\Gamma \otimes \mathbb{R}} &= \sum_{e \in E_x} p(e) {}_{\text{Hom}(\Gamma, \mathbb{R})} \langle \omega, \Phi_0(t(e)) - \Phi_0(o(e)) \rangle_{\Gamma \otimes \mathbb{R}} \\
&= \sum_{e \in E_x} p(e) \tilde{\omega}(e) \\
&= \sum_{e \in (E_0)_{\pi(x)}} p(e) \omega(e) \\
&= -(\delta_p \omega)(\pi(x)) \\
&= \langle \gamma_p, \omega \rangle = {}_{\text{Hom}(\Gamma, \mathbb{R})} \langle \omega, \rho_{\mathbb{R}}(\gamma_p) \rangle_{\Gamma \otimes \mathbb{R}} \quad (x \in V).
\end{aligned}$$

A periodic realization  $\Phi : X \rightarrow \Gamma \otimes \mathbb{R}$  satisfying (3.5) is said to be *modified harmonic*. Note that a modified harmonic realization is uniquely determined up to translation.

If we equip  $\Gamma \otimes \mathbb{R}$  with the Albanese metric  $g_0$ , then we call the map  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$  the *modified standard realization* of  $X$ . We readily check that the piecewise linear interpolation of  $\Phi_0$  by line segments descends to a piecewise geodesic map  $\Phi_0 : X_0 \rightarrow \text{Alb}^{\Gamma}$ . We call  $\Phi_0$  the *Albanese map* associated with  $(X, \Gamma)$ . Namely, standard realization is a lift of the Albanese map. In the symmetric case, Kotani–Sunada [11, 13] gave a characterization of the Albanese map  $(\Phi_0, g_0)$  through a “minimal principle” for the *energy*

$$\mathcal{E}(\Phi, g) := \frac{1}{2} \sum_{e \in E_0} |\Phi(t(\tilde{e})) - \Phi(o(\tilde{e}))|_g^2 \tilde{m}(e)$$

under the fixed volume condition  $\text{vol}(\Gamma \otimes \mathbb{R}/\Gamma \otimes \mathbb{Z}, g) = \text{vol}(\text{Alb}^{\Gamma})$ , where  $g$  is a flat metric on  $\Gamma \otimes \mathbb{R}/\Gamma \otimes \mathbb{Z}$ . In the forthcoming paper [4], we will discuss such a variational characterization of the Albanese map  $(\Phi_0, g_0)$  in the non-symmetric case.

## 3.2 Ergodic theorems

In this subsection, we make a preparation from ergodic theorems which will be used in the proof of main results. We set  $\ell^2(X_0) := \{f : V_0 \rightarrow \mathbb{C}\}$ , which is equipped with the inner product

$$\langle f_1, f_2 \rangle_{\ell^2(X_0)} := \sum_{x \in V_0} f_1(x) \overline{f_2(x)} \quad (f_1, f_2 \in \ell^2(X_0)).$$

Although the following ergodic theorem is standard, we give a proof for the sake of completeness.

**Theorem 3.2** *Let  $L$  be the transition operator on  $X_0$  associated with the transition probability  $p$ . Then it holds*

$$\frac{1}{N} \sum_{j=0}^{N-1} L^j f(x) = \sum_{x \in V_0} f(x) m(x) + O\left(\frac{1}{N}\right) \quad (x \in V_0, f \in \ell^2(X_0)).$$

**Proof.** We denote by  $K_0$  ( $1 \leq K_0 \leq |V_0|$ ) the period of the random walk on  $X_0$ , and set

$$\alpha_k := \exp\left(\frac{2\pi k}{K_0}\sqrt{-1}\right) \quad (k = 0, \dots, K_0 - 1),$$

where  $|V_0|$  stands for the number of elements of  $V_0$ . By virtue of the Perron–Frobenius theorem, the transition operator  $L : \ell^2(X_0) \rightarrow \ell^2(X_0)$  has the maximal simple eigenvalues  $\alpha_0, \dots, \alpha_{K_0-1}$  with the corresponding normalized right eigenfunctions  $\phi_0, \dots, \phi_{K_0-1}$  and left eigenfunctions  $\psi_0, \dots, \psi_{K_0-1}$ . Namely,

$$L\phi_k = \alpha_k\phi_k, \quad {}^tL\psi_k = \overline{\alpha_k}\psi_k, \quad \|\phi_k\|_{\ell^2(X_0)} = \langle \phi_k, \psi_k \rangle_{\ell^2(X_0)} = 1 \quad (k = 0, \dots, K_0 - 1).$$

In particular, we obtain  $\phi_0 \equiv |V_0|^{-1/2}$  and  $\psi_0(x) = |V_0|^{1/2}m(x)$  for  $x \in V_0$ .

Now we set

$$\ell_{K_0}^2(X_0) := \{f \in \ell^2(X_0) \mid \langle f, \psi_k \rangle_{\ell^2(X_0)} = 0 \quad (k = 0, \dots, K_0 - 1)\}.$$

Note that  $\ell_{K_0}^2(X_0)$  is preserved by  $L$ . Thus  $f \in \ell^2(X_0)$  is decomposed as

$$\begin{aligned} f &= \sum_{k=0}^{K_0-1} \langle f, \psi_k \rangle_{\ell^2(X_0)} \phi_k + f_{\ell_{K_0}^2(X_0)} \\ &= \langle f, m \rangle_{\ell^2(X_0)} + \sum_{k=1}^{K_0-1} \langle f, \psi_k \rangle_{\ell^2(X_0)} \phi_k + f_{\ell_{K_0}^2(X_0)} \quad (f_{\ell_{K_0}^2(X_0)} \in \ell_{K_0}^2(X_0)). \end{aligned} \quad (3.6)$$

Besides, it follows from the Perron–Frobenius theorem that there exists some  $0 \leq \lambda < 1$  such that  $\|L|_{\ell_{K_0}^2(X_0)}\| \leq \lambda$ . Furthermore by noting  $\sum_{j=0}^{K_0-1} \alpha_k^j = 0$  ( $k = 1, \dots, K_0 - 1$ ) and (3.6), we have

$$\begin{aligned} &\left\| \frac{1}{N} \sum_{j=0}^{N-1} L^j f - \langle f, m \rangle_{\ell^2(X_0)} \right\|_{\ell^2(X_0)} \\ &= \left\| \frac{1}{N} \sum_{k=1}^{K_0-1} \langle f, \psi_k \rangle_{\ell^2(X_0)} \left( \sum_{j=0}^{N-1} \alpha_k^j \right) \phi_k + \frac{1}{N} \sum_{j=0}^{N-1} L^j f_{\ell_{K_0}^2(X_0)} \right\|_{\ell^2(X_0)} \\ &\leq \frac{1}{N} \left\{ \sum_{k=1}^{K_0-1} |\langle f, \psi_k \rangle_{\ell^2(X_0)}| \left( \left| \sum_{j=0}^{N-1} \alpha_k^j \right| \right) \right\} + \left( \frac{1}{N} \sum_{j=0}^{N-1} \|L|_{\ell_{K_0}^2(X_0)}\|^j \right) \|f_{\ell_{K_0}^2(X_0)}\|_{\ell^2(X_0)} \\ &\leq \frac{K_0 - 1}{N} \left( \sum_{k=1}^{K_0-1} |\langle f, \psi_k \rangle_{\ell^2(X_0)}| \right) + \frac{1}{(1 - \lambda)N} \|f_{\ell_{K_0}^2(X_0)}\|_{\ell^2(X_0)} = O\left(\frac{1}{N}\right). \end{aligned}$$

Thus we complete the proof.  $\blacksquare$

In the proof of the CLT of the second kind (Theorem 2.3), the following ergodic theorem plays a crucial role.

**Theorem 3.3** *Let  $L_{(\varepsilon)}$  be the transition operator on  $X_0$  associated with the transition probability  $p_\varepsilon$  given in (2.5). Then there exists sufficiently small  $\varepsilon_0 > 0$  such that*

$$\frac{1}{N} \sum_{j=0}^{N-1} L_{(\varepsilon)}^j f(x) = \sum_{x \in V_0} f(x) m(x) + O_{\varepsilon_0} \left( \frac{1}{N} \right) \quad (x \in V_0, f \in \ell^2(X_0))$$

*holds for all  $0 \leq \varepsilon \leq \varepsilon_0$ .*

To prove this theorem, we need a fundamental perturbation theory of linear operators taken from Parry–Pollicott [18, Proposition 4.6]

**Proposition 3.4** *Let  $B(\mathcal{V})$  denote the set of linear operators on a Banach space  $\mathcal{V}$ . Assume  $\mathcal{L}_0 \in B(\mathcal{V})$  has a simple isolated eigenvalue  $\alpha_0$  with corresponding eigenvector  $\phi_0$ . Then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the following identities hold for all  $\mathcal{L} \in B(\mathcal{V})$  with  $\|\mathcal{L} - \mathcal{L}_0\|_{B(\mathcal{V})} < \delta$ :*

- (1)  $\mathcal{L}$  has a simple isolated eigenvalue  $\alpha(\mathcal{L})$  and the corresponding eigenvector  $\phi(\mathcal{L})$  with  $\alpha(\mathcal{L}_0) = \alpha$ ,  $\phi(\mathcal{L}_0) = \phi_0$ ,
- (2)  $\mathcal{L} \mapsto \alpha(\mathcal{L})$ ,  $\mathcal{L} \mapsto \phi(\mathcal{L})$  are analytic,
- (3)  $|\alpha(\mathcal{L}) - \alpha_0| < \varepsilon$  and  $\text{Spec}(\mathcal{L}) \setminus \{\alpha(\mathcal{L})\} \subset \{z \in \mathbb{C} : |z - \alpha_0| > \varepsilon\}$ .

**Proof of Theorem 3.3.** Let  $0 \leq \varepsilon < 1$ . It follows from  $p_\varepsilon(e) > 0$  ( $e \in E_0$ ) that

$$K_0 = \begin{cases} 1 & \text{if } X_0 \text{ is non-bipartite,} \\ 2 & \text{if } X_0 \text{ is bipartite.} \end{cases}$$

We only consider the bipartite case because the argument for the non-bipartite case is same. Let  $V_0 = A_0 \amalg A_1$  be the bipartition. By the Perron–Frobenius theorem, the transition operator  $L_{(\varepsilon)} : \ell^2(X_0) \rightarrow \ell^2(X_0)$  ( $0 \leq \varepsilon < 1$ ) has the maximal simple eigenvalues  $\alpha_0(\varepsilon) = 1$ ,  $\alpha_1(\varepsilon) = -1$  with the corresponding normalized right eigenfunctions

$$\phi_0(\varepsilon) \equiv |V_0|^{-1/2}, \quad \phi_1(\varepsilon)(x) = \begin{cases} |V_0|^{-1/2} & (x \in A_0), \\ -|V_0|^{-1/2} & (x \in A_1), \end{cases}$$

and left eigenfunctions

$$\psi_0(\varepsilon)(x) = |V_0|^{1/2} m(x) \quad (x \in V_0), \quad \psi_1(\varepsilon)(x) = \begin{cases} |V_0|^{1/2} m(x) & (x \in A_0), \\ -|V_0|^{1/2} m(x) & (x \in A_1). \end{cases}$$

Then we see that the subspace  $\mathcal{U} := \{f \in \ell^2(X_0) | \langle f, \psi_0 \rangle_{\ell^2(X_0)} = \langle f, \psi_1 \rangle_{\ell^2(X_0)} = 0\}$  is independent of  $\varepsilon$  and it is preserved by  $L_{(\varepsilon)}$ . By virtue of Proposition 3.4, there exist sufficiently small  $\varepsilon_0 > 0$  and  $0 \leq \lambda < 1$  such that  $\|L_{(\varepsilon)}|_{\mathcal{U}}\| \leq \lambda$  holds for all  $0 \leq \varepsilon \leq \varepsilon_0$ . Hence we obtain the desired conclusion by following the proof of Theorem 3.2. ■



### 3.3 Trotter's approximation theorem

In this subsection, we quickly review an approximation theorem due to Trotter [24]. Let  $\mathcal{V}$  and  $\mathcal{V}_N$  ( $N \in \mathbb{N}$ ) be Banach spaces and  $B(\mathcal{V}, \mathcal{V}_N)$  stands for the set of bounded linear operators from  $\mathcal{V}$  to  $\mathcal{V}_N$ . Suppose that there exist a constant  $C > 0$  and two families of operators  $\{P_N \in B(\mathcal{V}, \mathcal{V}_N)\}_{N=1}^\infty$  and  $\{U_N \in B(\mathcal{V}_N)\}_{N=1}^\infty$  satisfying

$$\sup_{N \in \mathbb{N}} \|P_N\|_{B(\mathcal{V}, \mathcal{V}_N)} + \sup_{N, n \in \mathbb{N}} \|U_N^n\|_{B(\mathcal{V}_N)} \leq C, \quad \lim_{N \rightarrow \infty} \|P_N u\|_{\mathcal{V}_N} = \|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}).$$

Trotter's approximation theorem is then presented as follows:

**Theorem 3.5 (cf. [24, 9, 13])** *Let  $\{e^{-tT}\}_{t \geq 0}$  be a continuous semigroup on  $\mathcal{V}$  with the infinitesimal generator  $T$  and  $\{\tau_N\}_{N=1}^\infty$  be a decreasing sequence satisfying  $\lim_{N \rightarrow \infty} \tau_N = 0$ . Assume that there exists a core  $D$  of  $T$  such that*

$$\lim_{N \rightarrow \infty} \|T_N P_N u - P_N T u\|_{\mathcal{V}_N} = 0 \quad (u \in D),$$

where  $T_N := (1/\tau_N)(I - U_N)$ . Then for any sequence  $\{k_N\}_{N=1}^\infty$  of non-negative integers satisfying  $k_N \tau_N \rightarrow t$ , we have

$$\lim_{N \rightarrow \infty} \|U_N^{k_N} P_N u - P_N e^{-tT} u\|_{\mathcal{V}_N} = 0 \quad (u \in D).$$

## 4 Proof of the CLT of the first kind

In the following, we write  $\omega[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} := {}_{\text{Hom}(\Gamma, \mathbb{R})} \langle \omega, \mathbf{x} \rangle_{\Gamma \otimes \mathbb{R}}$  ( $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ ,  $\mathbf{x} \in \Gamma \otimes \mathbb{R}$ ) and set

$$d\Phi_0(e) := \Phi_0(t(e)) - \Phi_0(o(e)) \quad (e \in E), \quad \|d\Phi_0\|_\infty := \max_{e \in E_0} |d\Phi_0(\tilde{e})|_{g_0}.$$

We take an orthonormal basis  $\{\omega_1, \dots, \omega_d\}$  of  $\text{Hom}(\Gamma, \mathbb{R}) (\subset H^1(X_0, \mathbb{R}) \cong \mathcal{H}^1(X_0))$ , and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$  denote its dual basis in  $\Gamma \otimes \mathbb{R}$ . Namely,  $\omega_i[\mathbf{v}_j]_{\Gamma \otimes \mathbb{R}} = \delta_{ij}$  ( $i, j = 1, \dots, d$ ). Note that  $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$  is an orthonormal basis of  $\Gamma \otimes \mathbb{R}$  with respect to the Albanese metric  $g_0$ . Then we have

$$\mathbf{x} = \sum_{i=1}^d \omega_i[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} \mathbf{v}_i, \quad |\mathbf{x}|_{g_0}^2 = \sum_{i=1}^d \omega_i[\mathbf{x}]_{\Gamma \otimes \mathbb{R}}^2 \quad (\mathbf{x} \in \Gamma \otimes \mathbb{R}).$$

For simplicity, we also write  $x_i := \omega_i[\mathbf{x}]_{\Gamma \otimes \mathbb{R}}$ ,  $\Phi_0(x)_i := \omega_i[\Phi_0(x)]_{\Gamma \otimes \mathbb{R}}$  ( $i = 1, \dots, d$ ,  $x \in V$ ) and identify  $\mathbf{x} \in \Gamma \otimes \mathbb{R}$  with  $(x_1, \dots, x_d) \in \mathbb{R}^d$ .

### 4.1 Proof of Theorem 2.1

To prove Theorem 2.1, we need the following lemma:

**Lemma 4.1** For any  $f \in C_0^\infty(\Gamma \otimes \mathbb{R})$ , as  $N \nearrow \infty$ ,  $\varepsilon \searrow 0$  and  $N^2\varepsilon \searrow 0$ , we have

$$\left\| \frac{1}{N\varepsilon^2} \left( I - \mathcal{L}_{\gamma_p}^N \right) \mathcal{P}_\varepsilon f - \mathcal{P}_\varepsilon \left( \frac{\Delta}{2} f \right) \right\|_\infty \rightarrow 0,$$

where  $\Delta$  is the (positive) Laplacian  $-\sum_{i=1}^d \left( \frac{\partial}{\partial x_i} \right)^2$  on  $\Gamma \otimes \mathbb{R}$  with the Albanese metric  $g_0$ .

**Proof.** First, we define  $A^N(\Phi_0)_{ij} : V \rightarrow \mathbb{R}$  ( $i, j = 1, \dots, d$ ,  $N \in \mathbb{N}$ ) by

$$\begin{aligned} A^N(\Phi_0)_{ij}(x) &:= \sum_{c \in \Omega_{x,N}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p))_i \\ &\quad \times (\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p))_j \quad (x \in V), \end{aligned}$$

where  $p(c) := p(e_1)p(e_2)\cdots p(e_N)$  for  $c = (e_1, e_2, \dots, e_N) \in \Omega_{x,N}(X)$ . Applying Taylor's expansion formula to  $f(\varepsilon(\Phi_0(t(c)) - \rho_{\mathbb{R}}(\mathbf{z} + N\gamma_p)))$  at  $\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z})) \in \Gamma \otimes \mathbb{R}$ , we have

$$\begin{aligned} &f(\varepsilon(\Phi_0(t(c)) - \rho_{\mathbb{R}}(\mathbf{z} + N\gamma_p))) - f(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))) \\ &= \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))) \left( \varepsilon(\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p)) \right)_i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))) \left( \varepsilon(\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p)) \right)_i \\ &\quad \times \left( \varepsilon(\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p)) \right)_j + O(N^3\varepsilon^3). \end{aligned}$$

This implies

$$\begin{aligned} &(I - \mathcal{L}_{\gamma_p}^N) \mathcal{P}_\varepsilon f(x, \mathbf{z}) \\ &= -\varepsilon \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))) \sum_{c \in \Omega_{x,N}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p))_i \\ &\quad - \frac{\varepsilon^2}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z}))) A^N(\Phi_0)_{ij}(x) + O(N^3\varepsilon^3). \end{aligned}$$

Recalling the modified harmonicity (3.5), we obtain

$$\begin{aligned} &\sum_{c \in \Omega_{x,N}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - N\rho_{\mathbb{R}}(\gamma_p)) \\ &= \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \left\{ \left( \sum_{e \in E_{t(c')}} p(e) (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p)) \right) \right. \\ &\quad \left. + (\Phi_0(t(c')) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p)) \right\} \\ &= \sum_{c' \in \Omega_{x,N-1}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p)) \\ &= \sum_{e \in E_x} p(e) (\Phi_0(t(e)) - \Phi_0(x) - \rho_{\mathbb{R}}(\gamma_p)) = \mathbf{0}. \end{aligned} \tag{4.1}$$

Next we define  $\mathcal{A}(\Phi_0)_{ij} : V_0 \rightarrow \mathbb{R}$  ( $i, j = 1, \dots, d$ ) by

$$\begin{aligned} \mathcal{A}(\Phi_0)_{ij}(x) &= \sum_{e \in (E_0)_x} p(e) (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) - \rho_{\mathbb{R}}(\gamma_p))_i \\ &\quad \times (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) - \rho_{\mathbb{R}}(\gamma_p))_j \quad (x \in V_0), \end{aligned}$$

where  $\tilde{e}$  is a lift of  $e \in E_0$  to  $E$ . Because  $A^N(\Phi_0)_{ij} : V \rightarrow \mathbb{R}$  is  $\Gamma$ -invariant, we easily see

$$\mathcal{A}(\Phi_0)_{ij}(\pi(x)) = A^1(\Phi_0)_{ij}(x) \quad (x \in V, i, j = 1, \dots, d).$$

Repeating the same argument as in (4.1) and recalling (3.5) again, we obtain

$$\begin{aligned} A^N(\Phi_0)_{ij}(x) &= \sum_{c' \in \Omega_{x, N-1}(X)} p(c') \sum_{e \in E_t(c')} p(e) \\ &\quad \times \left\{ (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i + (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_i \right\} \\ &\quad \times \left\{ (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_j + (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_j \right\} \\ &= \sum_{c' \in \Omega_{x, N-1}(X)} p(c') \sum_{e \in E_t(c')} p(e) \\ &\quad \times (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_j \\ &\quad + \sum_{c' \in \Omega_{x, N-1}(X)} p(c') \sum_{e \in E_t(c')} p(e) \\ &\quad \times \left\{ (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_j \right. \\ &\quad \left. + (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_j \right\} \\ &\quad + \sum_{c' \in \Omega_{x, N-1}(X)} p(c') \sum_{e \in E_t(c')} p(e) (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_i \\ &\quad \times (\Phi_0(o(e)) - \Phi_0(x) - (N-1)\rho_{\mathbb{R}}(\gamma_p))_j \\ &= L^{N-1}(\mathcal{A}(\Phi_0)_{ij})(\pi(x)) + A^{N-1}(\Phi_0)_{ij}(x) \\ &= \sum_{k=0}^{N-1} L^k(\mathcal{A}(\Phi_0)_{ij})(\pi(x)) \quad (x \in V). \end{aligned}$$

Applying Theorem 3.2, we have

$$\frac{1}{N} \sum_{k=0}^{N-1} L^k(\mathcal{A}(\Phi_0)_{ij})(\pi(x)) = \sum_{x \in V_0} \mathcal{A}(\Phi_0)_{ij}(x) m(x) + O\left(\frac{1}{N}\right).$$

Moreover (3.4) and (3.5) imply

$$\begin{aligned}
& \sum_{x \in X_0} \mathcal{A}(\Phi_0)_{ij}(x)m(x) \\
&= \sum_{e \in E_0} (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) - \rho_{\mathbb{R}}(\gamma_p))_i (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) - \rho_{\mathbb{R}}(\gamma_p))_j \tilde{m}(e) \\
&= \sum_{e \in E_0} (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})))_i (\Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})))_j \tilde{m}(e) - \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j \\
&= \sum_{e \in E_0} {}^t \rho_{\mathbb{R}}(\omega_i)(e) {}^t \rho_{\mathbb{R}}(\omega_j)(e) \tilde{m}(e) - \omega_i[\rho_{\mathbb{R}}(\gamma_p)]_{\Gamma \otimes \mathbb{R}} \omega_j[\rho_{\mathbb{R}}(\gamma_p)]_{\Gamma \otimes \mathbb{R}} \\
&= \sum_{e \in E_0} \omega_i(e) \omega_j(e) \tilde{m}(e) - \langle \gamma_p, \omega_i \rangle \langle \gamma_p, \omega_j \rangle \\
&= \langle \omega_i, \omega_j \rangle = \delta_{ij} \quad (i, j = 1, \dots, d).
\end{aligned}$$

Putting it all together, we now obtain

$$\frac{1}{N\varepsilon^2}(I - \mathcal{L}_{\gamma_p}^N)\mathcal{P}_\varepsilon f(x, \mathbf{z}) = \mathcal{P}_\varepsilon\left(\frac{\Delta}{2}f\right)(x, \mathbf{z}) + O(N^2\varepsilon) \quad \text{as } N \rightarrow \infty.$$

Finally by letting  $N^2\varepsilon \rightarrow 0$ , we complete the proof.  $\blacksquare$

**Proof of Theorem 2.1.** We follow the proof of [9, Theorem 4]. Let  $N = N(n)$  be the integer with  $n^{1/5} \leq N < 1 + n^{1/5}$  and  $k_N$  and  $r_N$  are the quotient and remainder of  $([nt] - [ns])/N$ , respectively. We put  $\varepsilon_N := n^{-1/2}$  and  $\tau_N := N\varepsilon_N^2$ . Then  $n \rightarrow \infty$  implies  $N \rightarrow \infty$ ,  $N^2\varepsilon_N \leq (1 + n^{1/5})^2 n^{-1/2} \rightarrow 0$  and  $\tau_N \leq (1 + n^{1/5})/n \rightarrow 0$ . As  $r_N < N$ , we also observe  $r_N\varepsilon_N^2 \leq N\varepsilon_N^2 \leq (1 + n^{1/5})/n \rightarrow 0$ . Noting  $k_N\tau_N = ([nt] - [ns] - r_N)\varepsilon_N^2$ , we obtain  $k_N\tau_N \rightarrow (t - s)$  as  $N \rightarrow \infty$ .

Now we may apply Theorem 3.5 to the case where

$$\mathcal{V} = C_\infty(\Gamma \otimes \mathbb{R}), \quad \mathcal{V}_N = C_\infty(X \times \mathbf{H}_1(X_0, \mathbb{R})), \quad U_N = \mathcal{L}_{\gamma_p}^N, \quad T = \frac{\Delta}{2}, \quad D = C_0^\infty(\Gamma \otimes \mathbb{R}),$$

and Lemma 4.1 implies

$$\lim_{n \rightarrow \infty} \left\| \mathcal{L}_{\gamma_p}^{k_N N} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{-\frac{t-s}{2}\Delta} f \right\|_\infty = 0 \quad (f \in C_0^\infty(\Gamma \otimes \mathbb{R})). \quad (4.2)$$

Further, we have

$$\begin{aligned}
& \left\| \mathcal{L}_{\gamma_p}^{[nt]-[ns]} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{-\frac{t-s}{2}\Delta} f \right\|_\infty \\
& \leq \left\| \mathcal{L}_{\gamma_p}^{k_N N} (\mathcal{L}_{\gamma_p}^{r_N} - I) \mathcal{P}_{n^{-1/2}} f \right\|_\infty + \left\| \mathcal{L}_{\gamma_p}^{k_N N} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{-\frac{t-s}{2}\Delta} f \right\|_\infty.
\end{aligned} \quad (4.3)$$

Noting  $r_N^2\varepsilon_N \leq (1 + n^{1/5})^2 n^{-1/2} \rightarrow 0$  and recalling Lemma 4.1 again, we obtain

$$\left\| \frac{1}{r_N\varepsilon_N^2} (I - \mathcal{L}_{\gamma_p}^{r_N}) \mathcal{P}_{\varepsilon_N} f - \mathcal{P}_{\varepsilon_N} \left( \frac{\Delta}{2} f \right) \right\|_\infty \rightarrow 0 \quad (f \in C_0^\infty(\Gamma \otimes \mathbb{R})).$$

This convergence and  $r_N \varepsilon_N^2 \rightarrow 0$  imply

$$\begin{aligned} \|\mathcal{L}_{\gamma_p}^{k_N N}(\mathcal{L}_{\gamma_p}^{r_N} - I)\mathcal{P}_{n^{-1/2}}f\|_\infty &\leq r_N \varepsilon_N^2 \left\| \frac{1}{r_N \varepsilon_N^2} (I - \mathcal{L}_{\gamma_p}^{r_N}) \mathcal{P}_{\varepsilon_N} f - \mathcal{P}_{\varepsilon_N} \left( \frac{\Delta}{2} f \right) \right\|_\infty \\ &\quad + r_N \varepsilon_N^2 \left\| \frac{\Delta}{2} f \right\|_\infty \rightarrow 0. \end{aligned} \quad (4.4)$$

Hence by combining (4.2), (4.3) with (4.4), we obtain (2.3) for  $f \in C_0^\infty(\Gamma \otimes \mathbb{R})$ .

For  $f \in C_\infty(\Gamma \otimes \mathbb{R})$ , we can choose a sequence  $\{f_m\}_{m=1}^\infty \subset C_0^\infty(\Gamma \otimes \mathbb{R})$  such that  $\|f - f_m\|_\infty \rightarrow 0$  as  $m \rightarrow \infty$ . Because

$$\max \left\{ \left\| \mathcal{L}_{\gamma_p}^{[nt] - [ns]} \mathcal{P}_{n^{-1/2}}(f - f_m) \right\|_\infty, \left\| \mathcal{P}_{n^{-1/2}} e^{-\frac{t-s}{2}\Delta} (f - f_m) \right\|_\infty \right\} \leq \|f - f_m\|_\infty$$

holds for each  $n \in \mathbb{N}$ , it is straightforward to check (2.3) for  $f \in C_\infty(\Gamma \otimes \mathbb{R})$ .

In addition, we have

$$\begin{aligned} &|\mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, \mathbf{z}_n) - e^{-t\Delta/2} f(\mathbf{x})| \\ &\leq \left\| \mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} e^{-\frac{t}{2}\Delta} f \right\|_\infty \\ &\quad + \left| e^{-\frac{t}{2}\Delta} f(n^{-1/2}(\Phi_0(x_n) - \rho_{\mathbb{R}}(\mathbf{z}_n))) - e^{-\frac{t}{2}\Delta} f(\mathbf{x}) \right| \end{aligned} \quad (4.5)$$

for  $f \in C_\infty(\Gamma \otimes \mathbb{R})$ . By combining the continuity of  $e^{-\frac{t}{2}\Delta} f : \Gamma \otimes \mathbb{R} \rightarrow \mathbb{R}$  and (2.3) with (4.5), we obtain (2.4). This completes the proof.  $\blacksquare$

## 4.2 Proof of Theorem 2.2

At the beginning, we show the convergence of finite dimensional distribution of  $\{\mathbf{X}^{(n)}\}_{n=1}^\infty$  by using Theorem 2.1. We fix  $0 \leq t_1 < \dots < t_r < \infty$  ( $r \in \mathbb{N}$ ) and consider the random variable  $\mathbf{X}_{t_1, t_2, \dots, t_r}^{(n)} : \Omega_{x_*}(X) \rightarrow (\Gamma \otimes \mathbb{R})^r$  given by

$$\mathbf{X}_{t_1, \dots, t_r}^{(n)}(c) := (\mathbf{X}_{t_1}^{(n)}(c), \dots, \mathbf{X}_{t_r}^{(n)}(c)).$$

### Lemma 4.2

$$\mathbf{X}_{t_1, \dots, t_r}^{(n)} \xrightarrow{\mathcal{D}} (B_{t_1}, \dots, B_{t_r}) \quad \text{as } n \rightarrow \infty,$$

where  $(B_t)_{t \geq 0}$  is a  $\Gamma \otimes \mathbb{R}$ -valued standard Brownian motion with  $B_0 = \mathbf{0}$ .

**Proof.** For simplicity, we only show the convergence for  $r = 2$ . General cases differ from this one only by being notationally more cumbersome. Take  $f_1 = f_1(\mathbf{y}_1), f_2 = f_2(\mathbf{y}_2) \in C_\infty(\Gamma \otimes \mathbb{R})$ , and set  $s = t_1, t = t_2$ . Let  $f(\mathbf{y}_1, \mathbf{y}_2) := f_1(\mathbf{y}_1)f_2(\mathbf{y}_2)$ . We note that  $\{f = f_1(\mathbf{y}_1)f_2(\mathbf{y}_2) \mid f_1, f_2 \in C_\infty(\Gamma \otimes \mathbb{R})\} \subset C_b((\Gamma \otimes \mathbb{R})^2)$  is a *determining class* on  $(\Gamma \otimes \mathbb{R})^2$ . (See e.g. Karatzas–Shreve [6, Definition 5.4.24] for the precise meaning. In Klenke [8, Definition 13.9], it is called a *separating family*.)

We aim to show

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{(\Gamma \otimes \mathbb{R})^2} f(\mathbf{y}_1, \mathbf{y}_2) (\mathbb{P}_{x_*} \circ (\mathbf{X}_{s,t}^{(n)})^{-1}) (d\mathbf{y}_1 d\mathbf{y}_2) \\ &= \int_{(\Gamma \otimes \mathbb{R})^2} G_s(\mathbf{y}_1) G_{t-s}(\mathbf{y}_2 - \mathbf{y}_1) f(\mathbf{y}_1, \mathbf{y}_2) d\mathbf{y}_1 d\mathbf{y}_2. \end{aligned} \quad (4.6)$$

We prepare

$$\begin{aligned} \sup_{c \in \Omega_{x_*}(X)} |\mathbf{X}_t^{(n)}(c) - \mathcal{X}_t^{(n)}(c)|_{g_0} &= \frac{(nt - [nt])}{\sqrt{n}} \sup_{c \in \Omega_{x_*}(X)} |\xi_{[nt]+1}(c) - \xi_{[nt]}(c) - \rho_{\mathbb{R}}(\gamma_p)|_{g_0} \\ &\leq \frac{1}{\sqrt{n}} (\|d\Phi_0\|_{\infty} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0}). \end{aligned} \quad (4.7)$$

Then, by virtue of uniform continuity of  $f$  on  $(\Gamma \otimes \mathbb{R})^2$  and (4.7), we obtain

$$\lim_{n \rightarrow \infty} \sum_{c \in \Omega_{x_*}(X)} |f(\mathbf{X}_{s,t}^{(n)}(c)) - f(\mathcal{X}_{s,t}^{(n)}(c))| \mathbb{P}_{x_*}(\{c\}) = 0.$$

Moreover, we have

$$\begin{aligned} & \sum_{c \in \Omega_{x_*}(X)} f(\mathcal{X}_{s,t}^{(n)}(c)) \mathbb{P}_{x_*}(\{c\}) \\ &= \sum_{c_1 \in \Omega_{x_*}, [ns](X)} p(c_1) f_1 \left( n^{-1/2} (\Phi_0(t(c_1)) - [ns] \rho_{\mathbb{R}}(\gamma_p)) \right) \\ & \quad \times \left\{ \sum_{c_2 \in \Omega_{t(c_1), [nt]-[ns]}(X)} p(c_2) f_2 \left( n^{-1/2} (\Phi_0(t(c_2)) - [nt] \rho_{\mathbb{R}}(\gamma_p)) \right) \right\} \\ &= \sum_{c_1 \in \Omega_{x_*}, [ns](X)} p(c_1) (\mathcal{P}_{n^{-1/2}} f_1)(t(c_1), [ns] \rho_{\mathbb{R}}(\gamma_p)) \mathcal{L}_{\gamma_p}^{[nt]-[ns]}(\mathcal{P}_{n^{-1/2}} f_2)(t(c_1), [ns] \rho_{\mathbb{R}}(\gamma_p)) \\ &= \mathcal{L}_{\gamma_p}^{[ns]} \{ (\mathcal{P}_{n^{-1/2}} f_1) \cdot \mathcal{L}_{\gamma_p}^{[nt]-[ns]}(\mathcal{P}_{n^{-1/2}} f_2) \} (x_*, \mathbf{0}). \end{aligned}$$

Then Theorem 2.1 implies

$$\begin{aligned} & \left\| \mathcal{L}_{\gamma_p}^{[ns]} \{ (\mathcal{P}_{n^{-1/2}} f_1) \cdot \mathcal{L}_{\gamma_p}^{[nt]-[ns]}(\mathcal{P}_{n^{-1/2}} f_2) \} - \mathcal{P}_{n^{-1/2}} e^{-\frac{s}{2}\Delta} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) \right\|_{\infty} \\ & \leq \left\| \mathcal{L}_{\gamma_p}^{[ns]} \{ (\mathcal{P}_{n^{-1/2}} f_1) \cdot \mathcal{L}_{\gamma_p}^{[nt]-[ns]}(\mathcal{P}_{n^{-1/2}} f_2) \} \right. \\ & \quad \left. - \mathcal{L}_{\gamma_p}^{[ns]} \{ (\mathcal{P}_{n^{-1/2}} f_1) \cdot \mathcal{P}_{n^{-1/2}} (e^{-\frac{t-s}{2}\Delta} f_2) \} \right\|_{\infty} \\ & \quad + \left\| \mathcal{L}_{\gamma_p}^{[ns]} \mathcal{P}_{n^{-1/2}} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) - \mathcal{P}_{n^{-1/2}} e^{-\frac{s}{2}\Delta} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) \right\|_{\infty} \\ & \leq \|f_1\|_{\infty} \left\| \mathcal{L}_{\gamma_p}^{[nt]-[ns]}(\mathcal{P}_{n^{-1/2}} f_2) - \mathcal{P}_{n^{-1/2}} (e^{-\frac{t-s}{2}\Delta} f_2) \right\|_{\infty} \\ & \quad + \left\| \mathcal{L}_{\gamma_p}^{[ns]} \mathcal{P}_{n^{-1/2}} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) - \mathcal{P}_{n^{-1/2}} e^{-\frac{s}{2}\Delta} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) \right\|_{\infty} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Summarizing all the above, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{(\Gamma \otimes \mathbb{R})^2} f(\mathbf{y}_1, \mathbf{y}_2) (\mathbb{P}_{x_*} \circ (\mathbf{X}_{s,t}^{(n)})^{-1}) (d\mathbf{y}_1 d\mathbf{y}_2) \\
&= \lim_{n \rightarrow \infty} \sum_{c \in \Omega_{x_*}(X)} f(\mathcal{X}_{s,t}^{(n)}(c)) \mathbb{P}_{x_*}(\{c\}) \\
&= \lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[ns]} \{ \mathcal{P}_{n^{-1/2}} f_1 \cdot \mathcal{P}_{n^{-1/2}} (e^{-\frac{t-s}{2}\Delta} f_2) \} (x_*, \mathbf{0}) \\
&= \lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[ns]} \mathcal{P}_{n^{-1/2}} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) (x_*, \mathbf{0}) \\
&= \lim_{n \rightarrow \infty} \mathcal{P}_{n^{-1/2}} e^{-\frac{s}{2}\Delta} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) (x_*, \mathbf{0}) \\
&= e^{-\frac{s}{2}\Delta} (f_1 e^{-\frac{t-s}{2}\Delta} f_2) (\mathbf{0}) \\
&= \int_{\Gamma \otimes \mathbb{R}} d\mathbf{y}_1 G_s(\mathbf{y}_1) f_1(\mathbf{y}_1) \int_{\Gamma \otimes \mathbb{R}} d\mathbf{y}_2 G_{t-s}(\mathbf{y}_2 - \mathbf{y}_1) f_2(\mathbf{y}_2) \\
&= \int_{(\Gamma \otimes \mathbb{R})^2} f(\mathbf{y}_1, \mathbf{y}_2) G_s(\mathbf{y}_1) G_{t-s}(\mathbf{y}_2 - \mathbf{y}_1) d\mathbf{y}_1 d\mathbf{y}_2.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
\mathbb{P}_{x_*}(|\mathbf{X}_{s,t}^{(n)}|_{(\Gamma \otimes \mathbb{R})^2} > R) &\leq \mathbb{P}_{x_*}(|\mathbf{X}_s^{(n)}|_{g_0} > \frac{R}{\sqrt{2}}) + \mathbb{P}_{x_*}(|\mathbf{X}_t^{(n)}|_{g_0} > \frac{R}{\sqrt{2}}) \\
&\leq \frac{4}{R^4} (\mathbb{E}^{\mathbb{P}_{x_*}} [|\mathbf{X}_s^{(n)}|_{g_0}^4] + \mathbb{E}^{\mathbb{P}_{x_*}} [|\mathbf{X}_t^{(n)}|_{g_0}^4]) \\
&\leq CR^{-4}(s^2 + t^2),
\end{aligned}$$

where we used (4.8) below and Chebyshev's inequality. This estimate implies tightness of  $\{\mathbb{P}_{x_0} \circ (\mathcal{X}_{s,t}^{(n)})^{-1}\}_{n=1}^\infty$  in probability measures on  $((\Gamma \otimes \mathbb{R})^2, \mathcal{B}((\Gamma \otimes \mathbb{R})^2))$ .

Finally, applying [8, Theorem 13.16], we have the desired convergence (4.6) for every  $f \in C_b((\Gamma \otimes \mathbb{R})^2)$ . This completes the proof.  $\blacksquare$

Having shown the convergence of finite dimensional distributions, we complete the proof of Theorem 2.2 by showing the following lemma:

**Lemma 4.3**  $\{\mathbf{P}^{(n)}\}_{n=1}^\infty$  is tight in  $(\mathbf{W}, \mathcal{B}(\mathbf{W}))$ .

**Proof.** From [6, Problem 4.11] (see also [8, Theorem 21.42]), it is sufficient to show that there exists some  $C > 0$  independent of  $n$  such that

$$\mathbb{E}^{\mathbb{P}_{x_*}} [|\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0}^4] \leq C|t - s|^2 \quad (0 \leq s \leq t, n \in \mathbb{N}). \quad (4.8)$$

We distinguish two cases:

$$\text{(I)} \quad t - s < n^{-1}, \quad \text{(II)} \quad t - s \geq n^{-1}.$$

First, we consider case **(I)**. If  $ns \geq [nt]$ ,

$$|\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0} \leq n^{1/2}(t-s) \{ |\xi_{[nt]+1} - \xi_{[nt]}|_{g_0} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \}.$$

On the other hand, if  $ns < [nt]$ ,

$$\begin{aligned} |\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0} &\leq \left| \frac{1 - (ns - [ns])}{n^{1/2}} (\xi_{[nt]} - \xi_{[nt]-1}) + \frac{nt - [nt]}{n^{1/2}} (\xi_{[nt]+1} - \xi_{[nt]}) - n^{1/2}(t-s) \rho_{\mathbb{R}}(\gamma_p) \right|_{g_0} \\ &\leq n^{1/2}(t-s) \{ |\xi_{[nt]+1} - \xi_{[nt]}|_{g_0} + |\xi_{[nt]} - \xi_{[nt]-1}|_{g_0} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \}, \end{aligned}$$

where we used  $[ns] = [nt] - 1$  and  $ns \leq [nt] \leq nt$  for the third line.

In either case, we have

$$\begin{aligned} |\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0} &\leq n^{1/2}(t-s) \{ |\xi_{[nt]+1} - \xi_{[nt]}|_{g_0} + |\xi_{[nt]} - \xi_{[nt]-1}|_{g_0} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \} \\ &\leq n^{1/2}(t-s) (2\|d\Phi_0\|_{\infty} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0}). \end{aligned} \quad (4.9)$$

Now we recall  $n^2(t-s)^2 < 1$ . Then (4.9) implies

$$\mathbb{E}^{\mathbb{P}_{x_*}} [|\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0}^4] \leq (2\|d\Phi_0\|_{\infty} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0})^4 (t-s)^2.$$

Hence, we obtain the desired estimate (4.8) for case **(I)**.

Next, we consider case **(II)**. Let  $\mathcal{F}$  be the fundamental domain in  $X$  containing  $x_* \in V$  and define  $\mathfrak{M}_i^l = \mathfrak{M}_i^l(\Phi_0, \rho_{\mathbb{R}}(\gamma_p))$  ( $i = 1, \dots, d, l = 1, 2, 3, 4$ ) by

$$\mathfrak{M}_i^l(x) := \sum_{e \in E_x} p(e) (d\Phi_0(e) - \rho_{\mathbb{R}}(\gamma_p))_i^l \quad (x \in V).$$

Note that  $\mathfrak{M}_i^l$  is  $\Gamma$ -invariant and

$$\|\mathfrak{M}_i^l\|_{\infty} \leq (\|d\Phi_0\|_{\infty} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0})^l \quad (i = 1, \dots, d).$$

Furthermore, the modified harmonicity (3.5) yields  $\mathfrak{M}_i^1 \equiv 0$  ( $i = 1, \dots, d$ ).

We start by giving a bound on  $\mathbb{E}^{\mathbb{P}_{x_*}} [|\mathcal{X}_{\frac{M}{n}}^{(n)} - \mathcal{X}_{\frac{N}{n}}^{(n)}|_{g_0}^4]$  ( $n \in \mathbb{N}, M \geq N \in \mathbb{N}$ ). Applying an elementary inequality  $(a_1 + \dots + a_d)^2 \leq d(a_1^2 + \dots + a_d^2)$ , we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_{x_*}} [|\mathcal{X}_{\frac{M}{n}}^{(n)} - \mathcal{X}_{\frac{N}{n}}^{(n)}|_{g_0}^4] &\leq dn^{-2} \sum_{i=1}^d \mathbb{E}^{\mathbb{P}_{x_*}} [(\xi_M - \xi_N - (M-N)\rho_{\mathbb{R}}(\gamma_p))_i^4] \\ &= d^2 n^{-2} \max_{i=1, \dots, d} \sum_{c_1 \in \Omega_{x_*, N}(X)} p(c_1) \sum_{c_2 \in \Omega_{t(c_1), M-N}(X)} p(c_2) \\ &\quad \times (\Phi_0(t(c_2)) - \Phi_0(t(c_1)) - (M-N)\rho_{\mathbb{R}}(\gamma_p))_i^4 \\ &= d^2 n^{-2} \max_{i=1, \dots, d} \max_{x \in \mathcal{F}} \left\{ \sum_{c \in \Omega_{x, M-N}(X)} p(c) \right. \\ &\quad \left. \times (\Phi_0(t(c)) - \Phi_0(x) - (M-N)\rho_{\mathbb{R}}(\gamma_p))_i^4 \right\}. \end{aligned} \quad (4.10)$$



We now fix  $i = 1, \dots, d$  and  $x \in V$ . For  $k = 1, \dots, M - N$ , we have

$$\begin{aligned}
& \sum_{c \in \Omega_{x,k}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - k\rho_{\mathbb{R}}(\gamma_p))_i^4 \\
&= \sum_{c' \in \Omega_{x,k-1}(X)} p(c') \sum_{e \in E_t(c')} p(e) \left\{ (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i \right. \\
&\quad \left. + (\Phi_0(o(e)) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i \right\}^4 \\
&= \sum_{c' \in \Omega_{x,k-1}(X)} p(c') \mathfrak{M}_i^4(t(c')) \\
&\quad + 4 \sum_{c' \in \Omega_{x,k-1}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i \mathfrak{M}_i^3(t(c')) \\
&\quad + 6 \sum_{c' \in \Omega_{x,k-1}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^2 \mathfrak{M}_i^2(t(c')) \\
&\quad + 4 \sum_{c' \in \Omega_{x,k-1}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^3 \mathfrak{M}_i^1(t(c')) \\
&\quad + \sum_{c' \in \Omega_{x,k-1}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^4 \\
&\leq \sum_{c \in \Omega_{x,k-1}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^4 \\
&\quad + \|\mathfrak{M}_i^4\|_{\infty} + 4 \{ \|d\Phi_0\|_{\infty} + (k-1)|\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \} \|\mathfrak{M}_i^3\|_{\infty} \\
&\quad + 6 \|\mathfrak{M}_i^2\|_{\infty} \sum_{c \in \Omega_{x,k-1}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^2. \tag{4.11}
\end{aligned}$$

Furthermore, it follows from the modified harmonicity (3.5) that

$$\begin{aligned}
& \sum_{c \in \Omega_{x,k-1}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i^2 \\
&= \sum_{c' \in \Omega_{x,k-2}(X)} p(c') \sum_{e \in E_t(c')} p(e) \left\{ (\Phi_0(t(e)) - \Phi_0(o(e)) - \rho_{\mathbb{R}}(\gamma_p))_i \right. \\
&\quad \left. + (\Phi_0(o(e)) - \Phi_0(x) - (k-1)\rho_{\mathbb{R}}(\gamma_p))_i \right\}^2 \\
&= \sum_{c' \in \Omega_{x,k-2}(X)} p(c') \left\{ (\Phi_0(t(c')) - \Phi_0(x) - (k-2)\rho_{\mathbb{R}}(\gamma_p))_i^2 + \mathfrak{M}_i^2(t(c')) \right\} \\
&\leq \sum_{c' \in \Omega_{x,k-2}(X)} p(c') (\Phi_0(t(c')) - \Phi_0(x) - (k-2)\rho_{\mathbb{R}}(\gamma_p))_i^2 + \|\mathfrak{M}_i^2\|_{\infty} \\
&\leq (k-1) \|\mathfrak{M}_i^2\|_{\infty}. \tag{4.12}
\end{aligned}$$

Combining (4.11) with (4.12), we obtain

$$\begin{aligned} & \sum_{c \in \Omega_{x,k}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - k\rho_{\mathbb{R}}(\gamma_p))_i^4 \\ & \leq (k-1) \left( 4|\rho_{\mathbb{R}}(\gamma_p)|_{\Gamma \otimes \mathbb{R}} \|\mathfrak{M}_i^3\|_{\infty} + 6\|\mathfrak{M}_i^2\|_{\infty}^2 \right) + (\|\mathfrak{M}_i^4\|_{\infty} + 6\|d\Phi_0\|_{\infty} \|\mathfrak{M}_i^3\|_{\infty}) = Ck, \end{aligned}$$

and it implies

$$\sum_{c \in \Omega_{x,M-N}(X)} p(c) (\Phi_0(t(c)) - \Phi_0(x) - (M-N)\rho_{\mathbb{R}}(\gamma_p))_i^4 \leq C(M-N)^2. \quad (4.13)$$

Therefore, by combining (4.10) with (4.13), we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_{x*}} [|\mathbf{X}_t^{(n)} - \mathbf{X}_s^{(n)}|_{g_0}^4] & \leq \mathbb{E}^{\mathbb{P}_{x*}} \left[ \max_{i,j=0,1} |\mathcal{X}_{\frac{[nt]+i}{n}}^{(n)} - \mathcal{X}_{\frac{[ns]+j}{n}}^{(n)}|_{g_0}^4 \right] \\ & \leq Cd^2 n^{-2} ([nt] - [ns] + 1)^2 \\ & \leq Cd^2 \left( t - s + \frac{2}{n} \right)^2 \\ & \leq Cd^2 \{ (t-s) + 2(t-s) \}^2 = C(t-s)^2, \end{aligned}$$

where we used  $[nt] - [ns] \leq n(t-s) + 1$  for the third line and the condition  $n^{-1} \leq (t-s)$  for the final line. Thus we obtained our desired estimate (4.8) for case **(II)**. ■

## 5 Proof of the CLT of the second kind

### 5.1 Basic properties of the family of Albanese metrics $\{g_0^{(\varepsilon)}\}$

Let  $\{p_{\varepsilon}\}_{0 \leq \varepsilon \leq 1}$  be a family of transition probabilities defined by (2.5). Thanks to [14, Proposition 2.3], we easily obtain the following lemma:

**Lemma 5.1** (1)  $p_1 = p$  and  $\gamma_{p_{\varepsilon}} = \varepsilon\gamma_p$  for every  $0 \leq \varepsilon \leq 1$ .

(2)  $p_{\varepsilon}(e) > 0$  for every  $e \in E_0$  provided  $0 \leq \varepsilon < 1$ .

(3) For every  $0 \leq \varepsilon \leq 1$ , the invariant measure of the random walk given by  $p_{\varepsilon}$  is  $m$ . Moreover,  $p_0$  and  $q$  are  $(m-)$ symmetric and  $(m-)$ anti-symmetric, respectively. Specifically,

$$p_0(e)m(o(e)) = p_0(\bar{e})m(t(e)), \quad q(e)m(o(e)) = -q(\bar{e})m(t(e)) \quad (e \in E_0).$$

(4)

$$\sum_{e \in E_0} q(e)\omega_1(e)\omega_2(e)m(o(e)) = 0 \quad (\omega_1, \omega_2 \in C^1(X_0, \mathbb{R})). \quad (5.1)$$

We put  $\tilde{m}_{\varepsilon}(e) := p_{\varepsilon}(e)m(o(e))$  ( $0 \leq \varepsilon \leq 1$ ,  $e \in E_0$ ) and denote by  $\mathcal{H}_{(\varepsilon)}^1(X_0)$  the set of modified harmonic 1-forms with respect to the transition probability  $p_{\varepsilon}$ . Namely,  $\mathcal{H}_{(\varepsilon)}^1(X_0)$  is the set of 1-forms on  $X_0$  satisfying

$$(\delta_{p_{\varepsilon}}\omega)(x) + \langle \gamma_{p_{\varepsilon}}, \omega \rangle = 0 \quad (x \in V_0).$$

We equip  $\mathcal{H}_{(\varepsilon)}^1(X_0)$  with the inner product

$$\begin{aligned}\langle\langle\omega_1, \omega_2\rangle\rangle_{(\varepsilon)} &:= \sum_{e \in E_0} \omega_1(e) \omega_2(e) \tilde{m}_\varepsilon(e) - \langle\gamma_{p_\varepsilon}, \omega_1\rangle \langle\gamma_{p_\varepsilon}, \omega_2\rangle \\ &= \sum_{e \in E_0} \omega_1(e) \omega_2(e) \tilde{m}_\varepsilon(e) - \varepsilon^2 \langle\gamma_p, \omega_1\rangle \langle\gamma_p, \omega_2\rangle \quad (\omega_1, \omega_2 \in \mathcal{H}_{(\varepsilon)}^1(X_0)),\end{aligned}$$

The corresponding norm  $\|\cdot\|_{(\varepsilon)}$  is given by

$$\|\omega\|_{(\varepsilon)}^2 := \langle\langle\omega, \omega\rangle\rangle_{(\varepsilon)} = \sum_{e \in E_0} |\omega(e)|^2 \tilde{m}_\varepsilon(e) - \varepsilon^2 \langle\gamma_p, \omega\rangle^2 \quad (\omega \in \mathcal{H}_{(\varepsilon)}^1(X_0)).$$

By the discrete Hodge–Kodaira theorem mentioned in Section 3, we may identify  $H^1(X_0, \mathbb{R})$  with  $\mathcal{H}_{(\varepsilon)}^1(X_0)$  for each  $0 \leq \varepsilon \leq 1$ . (It should be noted that the identification map depends on the parameter  $\varepsilon$  and  $\mathcal{H}_{(1)}^1(X_0) = \mathcal{H}^1(X_0)$ .) Moreover, we also identify  $\text{Hom}(\Gamma, \mathbb{R})$  with  $\text{Image}({}^t\rho_{\mathbb{R}}) \subset H^1(X_0, \mathbb{R})$ . Hence we may regard  $\text{Hom}(\Gamma, \mathbb{R})$  as a subspace of  $\mathcal{H}_{(\varepsilon)}^1(X_0)$ . For  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ , we denote  ${}^t\rho_{\mathbb{R}}(\omega) \in H^1(X_0, \mathbb{R}) \cong \mathcal{H}_{(\varepsilon)}^1(X_0)$  by  $\omega^{(\varepsilon)}$ . Then we have

**Lemma 5.2**

$$\lim_{\varepsilon \searrow 0} \langle\langle\omega^{(\varepsilon)}, \eta^{(\varepsilon)}\rangle\rangle_{(\varepsilon)} = \langle\langle\omega^{(0)}, \eta^{(0)}\rangle\rangle_{(0)} \quad (\omega, \eta \in \text{Hom}(\Gamma, \mathbb{R})).$$

**Proof.** By following the proof of [14, Lemma 5.2], we observe that for any  $0 \leq \varepsilon \leq 1$ , there exist functions  $f^{(\varepsilon)}, g^{(\varepsilon)}$  defined on  $X_0$  such that

$$\omega^{(\varepsilon)} = \omega^{(0)} + df^{(\varepsilon)}, \quad \eta^{(\varepsilon)} = \eta^{(0)} + dg^{(\varepsilon)} (\in \mathcal{H}_{(\varepsilon)}^1(X_0)).$$

Since  $\omega^{(0)} \in \mathcal{H}_{(0)}^1(X_0)$ , we have  $\delta_{p_0}\omega^{(0)} = 0$ . Hence

$$\sum_{e \in E_0} p_0(e) \omega^{(0)}(e) df(e) m(o(e)) = 2 \langle \delta_{p_0} \omega^{(0)}, fm \rangle_0 = 0 \quad (f \in C^0(X_0, \mathbb{R})). \quad (5.2)$$

Combining (5.1), (5.2) with the identity  $\langle\gamma_p, df\rangle = 0$  ( $f \in C^0(X_0, \mathbb{R})$ ), we can expand

$$\begin{aligned}\langle\langle\omega^{(\varepsilon)}, \eta^{(\varepsilon)}\rangle\rangle_{(\varepsilon)} &= \sum_{e \in E_0} p_\varepsilon(e) (\omega^{(0)}(e) + df^{(\varepsilon)}(e)) (\eta^{(0)}(e) + dg^{(\varepsilon)}(e)) m(o(e)) \\ &\quad - \varepsilon^2 \langle\gamma_p, \omega^{(0)} + df^{(\varepsilon)}\rangle \langle\gamma_p, \eta^{(0)} + dg^{(\varepsilon)}\rangle \\ &= \langle\langle\omega^{(0)}, \eta^{(0)}\rangle\rangle_{(0)} + \langle\langle df^{(\varepsilon)}, dg^{(\varepsilon)}\rangle\rangle_{(0)} - \varepsilon^2 \langle\gamma_p, \omega^{(0)}\rangle \langle\gamma_p, \eta^{(0)}\rangle.\end{aligned}$$

Hence it suffices to show that  $\langle\langle df^{(\varepsilon)}, dg^{(\varepsilon)}\rangle\rangle_{(0)}$  converges to 0 as  $\varepsilon \searrow 0$ .

We now define the operator  $Q : \ell^2(X_0) \rightarrow \ell^2(X_0)$  by

$$Qf(x) := \sum_{e \in (E_0)_x} q(e) f(t(e)) \quad (x \in V_0).$$

Then the transition operator  $L_{(\varepsilon)} : \ell^2(X_0) \rightarrow \ell^2(X_0)$  associated with the transition probability  $p_\varepsilon$  is decomposed by  $L_{(\varepsilon)} = L_{(0)} + \varepsilon Q$ . We also note that anti-symmetry of  $q$  implies

$$\langle Qf, fm \rangle_{\ell^2(X_0)} = \langle Qf, fm \rangle_0 = 0 \quad (f \in \ell^2(X_0)). \quad (5.3)$$

Then (5.3) and the identity  $I - L_{(\varepsilon)} = \delta_{p_\varepsilon} d$  imply

$$\begin{aligned} \langle df^{(\varepsilon)}, df^{(\varepsilon)} \rangle_{(0)} &= \langle \langle d(f^{(\varepsilon)} - m(f^{(\varepsilon)})), d(f^{(\varepsilon)} - m(f^{(\varepsilon)})) \rangle \rangle_{(0)} \\ &= 2 \langle (I - L_{(0)})(f^{(\varepsilon)} - m(f^{(\varepsilon)})), (f^{(\varepsilon)} - m(f^{(\varepsilon)}))m \rangle_0 \\ &= 2 \langle (I - L_{(\varepsilon)})f^{(\varepsilon)}, (f^{(\varepsilon)} - m(f^{(\varepsilon)}))m \rangle_0 \\ &= 2 \langle \delta_{p_\varepsilon}(df^{(\varepsilon)}), (f^{(\varepsilon)} - m(f^{(\varepsilon)}))m \rangle_0, \end{aligned} \quad (5.4)$$

where

$$m(f^{(\varepsilon)}) := \sum_{x \in X_0} f^{(\varepsilon)}(x)m(x)$$

and we used  $L_{(\varepsilon)}m(f^{(\varepsilon)}) = m(f^{(\varepsilon)})$  for the third line.

Recalling  $\omega^{(0)} + df^{(\varepsilon)} \in \mathcal{H}_{(\varepsilon)}^1(X_0)$  and  $\delta_{p_0}\omega^{(0)} = 0$ , we have

$$\begin{aligned} \delta_{p_\varepsilon}(df^{(\varepsilon)})(x) &= -(\delta_{p_\varepsilon}\omega^{(0)})(x) - \varepsilon \langle \gamma_p, \omega^{(0)} \rangle \\ &= -\varepsilon((\delta_q\omega^{(0)})(x) + \langle \gamma_p, \omega^{(0)} \rangle) \quad (x \in V_0). \end{aligned} \quad (5.5)$$

Thus

$$f^{(\varepsilon)}(x) = L_{(\varepsilon)}f^{(\varepsilon)}(x) - \varepsilon((\delta_q\omega^{(0)})(x) + \langle \gamma_p, \omega^{(0)} \rangle) \quad (x \in V_0), \quad (5.6)$$

and (5.6) still holds if we replace  $f^{(\varepsilon)}$  by  $f^{(\varepsilon)} - m(f^{(\varepsilon)})$ . Namely,

$$\begin{aligned} f^{(\varepsilon)}(x) - m(f^{(\varepsilon)}) &= L_{(\varepsilon)}(f^{(\varepsilon)} - m(f^{(\varepsilon)}))(x) \\ &\quad - \varepsilon((\delta_q\omega^{(0)})(x) + \langle \gamma_p, \omega^{(0)} \rangle) \quad (x \in V_0). \end{aligned} \quad (5.7)$$

We now recall  $\ell^2(X_0) = \langle \phi_0 \rangle \oplus \ell_1^2(X_0)$ , where  $\phi_0$  and  $\ell_1^2(X_0)$  are introduced in the proof of Proposition 3.2. Note  $f^{(\varepsilon)} - m(f^{(\varepsilon)}) \in \ell_1^2(X_0)$  and that the transition operator  $L_{(\varepsilon)}$  maps  $\ell_1^2(X_0)$  to  $\ell_1^2(X_0)$  for all  $0 \leq \varepsilon \leq 1$ . Furthermore, as  $\alpha_0(\varepsilon) = 1$  is a simple eigenvalue of  $L_{(\varepsilon)}$  for all  $0 \leq \varepsilon \leq 1$ , the inverse operator of  $(1 - L_{(\varepsilon)})|_{\ell_1^2(X_0)} : \ell_1^2(X_0) \rightarrow \ell_1^2(X_0)$  exists. Because of  $\delta_q\omega^{(0)} + \langle \gamma_p, \omega^{(0)} \rangle \in \ell_1^2(X_0)$ , we can solve equation (5.7) as

$$f^{(\varepsilon)} - m(f^{(\varepsilon)}) = -\varepsilon(1 - L_{(\varepsilon)})|_{\ell_1^2(X_0)}^{-1}(\delta_q\omega^{(0)} + \langle \gamma_p, \omega^{(0)} \rangle). \quad (5.8)$$

Combining (5.8) with the identity

$$(1 - L_{(\varepsilon)})|_{\ell_1^2(X_0)}^{-1} = (1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1} \left[ 1 - \varepsilon Q|_{\ell_1^2(X_0)}(1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1} \right]^{-1},$$

we obtain

$$\begin{aligned}
\|f^{(\varepsilon)} - m(f^{(\varepsilon)})\|_{\ell^2(X_0)} &\leq \varepsilon \|(1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1}\| \left( \|\delta_q \omega^{(0)}\|_{\ell^2(X_0)} + |\langle \gamma_p, \omega^{(0)} \rangle| \right) \\
&\leq \varepsilon \|(1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1}\| \left( 1 - \varepsilon \|Q|_{\ell_1^2(X_0)}(1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1}\| \right)^{-1} \\
&\quad \times \left( \|\delta_q \omega^{(0)}\|_{\ell^2(X_0)} + |\langle \gamma_p, \omega^{(0)} \rangle| \right). \tag{5.9}
\end{aligned}$$

Now we choose a sufficiently small constant  $\varepsilon_0 > 0$  such that

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \left( 1 - \varepsilon \|Q|_{\ell_1^2(X_0)}(1 - L_{(0)})|_{\ell_1^2(X_0)}^{-1}\| \right)^{-1} \leq 2.$$

Then (5.9) implies

$$\sup_{0 \leq \varepsilon \leq \varepsilon_0} \|f^{(\varepsilon)} - m(f^{(\varepsilon)})\|_{\ell^2(X_0)} < \infty. \tag{5.10}$$

Finally, by combining (5.4), (5.5) with (5.10), we obtain  $\lim_{\varepsilon \searrow 0} \langle\langle df^{(\varepsilon)}, df^{(\varepsilon)} \rangle\rangle_{(0)} = 0$ . This completes the proof.  $\blacksquare$

We define the Albanese metric  $g_0^{(\varepsilon)}$  on  $\Gamma \otimes \mathbb{R}$  by the dual metric of  $\langle\langle \cdot, \cdot \rangle\rangle_{(\varepsilon)}$ . The following lemma on the family of Albanese metrics  $\{g_0^{(\varepsilon)}\}_{0 \leq \varepsilon \leq 1}$  plays a key role in the proof of Theorems 2.3 and 2.4.

**Lemma 5.3** (1)

$$\lim_{\varepsilon \searrow 0} \langle \mathbf{x}, \mathbf{y} \rangle_{g_0^{(\varepsilon)}} = \langle \mathbf{x}, \mathbf{y} \rangle_{g_0^{(0)}} \quad (\mathbf{x}, \mathbf{y} \in \Gamma \otimes \mathbb{R}).$$

(2) Let  $\Phi_0^{(\varepsilon)} : X \rightarrow \Gamma \otimes \mathbb{R}$  be the modified harmonic realization with respect to  $p_\varepsilon$ . Then we have

$$\lim_{\varepsilon \searrow 0} \sum_{e \in E_0} |d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(\varepsilon)}}^2 \tilde{m}_\varepsilon(e) = \sum_{e \in E_0} |d\Phi_0^{(0)}(\tilde{e})|_{g_0^{(0)}}^2 \tilde{m}_0(e).$$

(3) Let  $\|d\Phi_0^{(\varepsilon)}\|_\infty := \max_{e \in E_0} |d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(0)}}$ , where  $\tilde{e}$  is a lift of  $e \in E_0$  to  $E$ . Then there exists a sufficiently small  $\varepsilon_0 > 0$  such that

$$\|d\Phi_0^{(\varepsilon)}\|_\infty \leq 2 \left( \min_{e \in E_0} \tilde{m}_0(e) \right)^{-1/2} \|d\Phi_0^{(0)}\|_\infty$$

for all  $0 \leq \varepsilon \leq \varepsilon_0$ .

**Proof.** (1) We first take an orthonormal basis  $\{\omega_1(\varepsilon), \dots, \omega_d(\varepsilon)\}$  of  $\text{Hom}(\Gamma, \mathbb{R}) (\subset \mathcal{H}_{(\varepsilon)}^1(X_0))$ . In particular, we write  $\omega_i = \omega_i(0)$  ( $i = 1, \dots, d$ ). Using the basis  $\{\omega_1, \dots, \omega_d\}$ , we may expand  $\omega_i(\varepsilon)$  as

$$\omega_i(\varepsilon) = \sum_{j=1}^d a(\varepsilon)_i^j \omega_j \quad (i = 1, \dots, d).$$

It follows from  $\langle\langle \omega_i(\varepsilon)^{(\varepsilon)}, \omega_j(\varepsilon)^{(\varepsilon)} \rangle\rangle_{(\varepsilon)} = \delta_{ij}$  that

$$\sum_{k,l=1}^d a(\varepsilon)_i^k a(\varepsilon)_j^l \langle\langle \omega_k^{(\varepsilon)}, \omega_l^{(\varepsilon)} \rangle\rangle_{(\varepsilon)} = \delta_{ij} \quad (i, j = 1, \dots, d).$$

Taking  $\varepsilon \searrow 0$  and using Lemma 5.2, we obtain

$$\lim_{\varepsilon \searrow 0} \sum_{k=1}^d a(\varepsilon)_i^k a(\varepsilon)_j^k = \delta_{ij} \quad (i, j = 1, \dots, d). \quad (5.11)$$

Now we set  $A(\varepsilon) = (a(\varepsilon)_i^j)_{i,j=1}^d \in \mathbb{R}^d \otimes \mathbb{R}^d$ . Since  $\omega_1(\varepsilon), \dots, \omega_d(\varepsilon)$  are linearly independent,  $A(\varepsilon)$  is invertible for every  $\varepsilon \geq 0$ . Moreover, we can rewrite (5.11) as

$$\lim_{\varepsilon \searrow 0} {}^t A(\varepsilon) A(\varepsilon) = I_{\mathbb{R}^d}.$$

This convergence yields  $\frac{1}{2} \leq \|A(\varepsilon)\| \leq 2$  for sufficiently small  $\varepsilon > 0$ . Then we have

$$\begin{aligned} \|A(\varepsilon) {}^t A(\varepsilon) - I_{\mathbb{R}^d}\| &= \|A(\varepsilon) ({}^t A(\varepsilon) A(\varepsilon) - I_{\mathbb{R}^d}) A(\varepsilon)^{-1}\| \\ &\leq 4 \|{}^t A(\varepsilon) A(\varepsilon) - I_{\mathbb{R}^d}\| \rightarrow 0 \quad \text{as } \varepsilon \searrow 0. \end{aligned}$$

It means

$$\lim_{\varepsilon \searrow 0} \sum_{i=1}^d a(\varepsilon)_i^k a(\varepsilon)_i^l = \delta^{kl} \quad (k, l = 1, \dots, d). \quad (5.12)$$

From (5.12), we have

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \langle \mathbf{x}, \mathbf{y} \rangle_{g_0^{(\varepsilon)}} &= \lim_{\varepsilon \searrow 0} \sum_{i=1}^d \omega_i(\varepsilon)[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} \omega_i(\varepsilon)[\mathbf{y}]_{\Gamma \otimes \mathbb{R}} \\ &= \sum_{k,l=1}^d \left( \lim_{\varepsilon \searrow 0} \sum_{i=1}^d a(\varepsilon)_i^k a(\varepsilon)_i^l \right) \omega_k[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} \omega_l[\mathbf{y}]_{\Gamma \otimes \mathbb{R}} \\ &= \sum_{k=1}^d \omega_k[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} \omega_k[\mathbf{y}]_{\Gamma \otimes \mathbb{R}} = \langle \mathbf{x}, \mathbf{y} \rangle_{g_0^{(0)}}. \end{aligned} \quad (5.13)$$

Thus we have shown (1). We also mention that (5.12) and (5.13) imply

$$\frac{1}{2} |\mathbf{x}|_{g_0^{(0)}} \leq |\mathbf{x}|_{g_0^{(\varepsilon)}} \leq 2 |\mathbf{x}|_{g_0^{(0)}} \quad (\mathbf{x} \in \Gamma \otimes \mathbb{R})$$

for sufficiently small  $\varepsilon \geq 0$ .

(2) Due to the identity (3.4), we can expand  $|d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(\varepsilon)}}^2$  as

$$\begin{aligned} |d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(\varepsilon)}}^2 &= \sum_{i=1}^d \omega_i(\varepsilon) [\Phi_0^{(\varepsilon)}(t(\tilde{e})) - \Phi_0^{(\varepsilon)}(o(\tilde{e}))]_{\Gamma \otimes \mathbb{R}}^2 \\ &= \sum_{i=1}^d \omega_i(\varepsilon)^{(e)}(e)^2 = \sum_{j,k=1}^d \left( \sum_{i=1}^d a_i^j(\varepsilon) a_i^k(\varepsilon) \right) \omega_j^{(\varepsilon)}(e) \omega_k^{(\varepsilon)}(e). \end{aligned} \quad (5.14)$$

Then by combining Lemma 5.2 and (5.12) with (5.14), we obtain

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \sum_{e \in E_0} p_\varepsilon(e) |d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(\varepsilon)}}^2 m(e) &= \lim_{\varepsilon \searrow 0} \sum_{j,k=1}^d \left( \sum_{i=1}^d a_i^j(\varepsilon) a_i^k(\varepsilon) \right) \langle \omega_j^{(\varepsilon)}, \omega_k^{(\varepsilon)} \rangle_{(\varepsilon)} \\ &= \sum_{j=1}^d \|\omega_j^{(0)}\|_{(0)}^2 = \sum_{e \in E_0} p_0(e) |d\Phi_0^{(0)}(\tilde{e})|_{g_0^{(0)}}^2 m(e). \end{aligned}$$

(3) Recall that  $\tilde{m}_\varepsilon(e)$  is continuous with respect to  $\varepsilon$  and  $\tilde{m}_0(e) > 0$  for all  $e \in E_0$ . Then by (5.13) and (5.14), we can choose a sufficiently small  $\varepsilon_0 > 0$  such that

$$\begin{aligned} |d\Phi_0^{(\varepsilon)}(e')|_{g_0^{(0)}}^2 &\leq 2 |d\Phi_0^{(\varepsilon)}(e')|_{g_0^{(\varepsilon)}}^2 \\ &\leq 2 \left( \min_{e \in E_0} \tilde{m}_\varepsilon(e) \right)^{-1/2} \sum_{e \in E_0} |d\Phi_0^{(\varepsilon)}(\tilde{e})|_{g_0^{(\varepsilon)}}^2 \tilde{m}_\varepsilon(e) \\ &\leq 4 \left( \min_{e \in E_0} \tilde{m}_0(e) \right)^{-1/2} \sum_{e \in E_0} |d\Phi_0^{(0)}(\tilde{e})|_{g_0^{(0)}}^2 \tilde{m}_0(e) \\ &\leq 4 \left( \min_{e \in E_0} \tilde{m}_0(e) \right)^{-1/2} \|d\Phi_0^{(0)}\|_\infty \end{aligned}$$

for all  $e' \in E_0$  and  $0 \leq \varepsilon \leq \varepsilon_0$ . This completes the proof.  $\blacksquare$

## 5.2 Proof of Theorem 2.3

To prove Theorem 2.3, the following lemma is essential.

**Lemma 5.4** *For any  $f \in C_0^\infty((\Gamma \otimes \mathbb{R})_{(0)})$ , as  $N \nearrow \infty$ ,  $\varepsilon \searrow 0$  and  $N^2\varepsilon \searrow 0$ , we have*

$$\left\| \frac{1}{N\varepsilon^2} (I - L_{(\varepsilon)}^N) P_\varepsilon f - P_\varepsilon \left( \frac{1}{2} \Delta_{(0)} - \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} \rangle_{g_0^{(0)}} \right) f \right\|_\infty \rightarrow 0,$$

where  $\Delta_{(0)}$  and  $\nabla_{(0)}$  stand for the (positive) Laplacian and the gradient on  $(\Gamma \otimes \mathbb{R})_{(0)}$ , respectively.

**Proof.** We first take an orthonormal basis  $\{\omega_1, \dots, \omega_d\}$  of  $\text{Hom}(\Gamma, \mathbb{R}) (\subset H^1(X_0, \mathbb{R}) \cong \mathcal{H}_{(0)}^1(X_0))$ , and let  $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$  denote its dual basis in  $\Gamma \otimes \mathbb{R}$ . Note that  $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$  is an orthonormal basis of  $(\Gamma \otimes \mathbb{R})_{(0)}$ . As in the previous section, we denote  $\omega_i[\mathbf{x}]_{\Gamma \otimes \mathbb{R}}$  by  $x_i$ , and identify  $\mathbf{x} \in \Gamma \otimes \mathbb{R}$  with  $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ . Applying Taylor's expansion formula, we have

$$\begin{aligned} &f(\varepsilon\Phi_0^{(\varepsilon)}(t(c))) - f(\varepsilon\Phi_0^{(\varepsilon)}(x)) \\ &= \varepsilon \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\varepsilon\Phi_0^{(\varepsilon)}(x)) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_i \\ &\quad + \frac{\varepsilon^2}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon\Phi_0^{(\varepsilon)}(x)) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_i (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_j + O(N^3\varepsilon^3). \end{aligned}$$

Recalling that the modified harmonicity of  $\Phi_0^{(\varepsilon)}$  and  $\rho_{\mathbb{R}}(\gamma_{p_\varepsilon}) = \varepsilon \rho_{\mathbb{R}}(\gamma_p)$ , we see

$$L_{(\varepsilon)}^N \Phi_0^{(\varepsilon)} = \Phi_0^{(\varepsilon)} + N \varepsilon \rho_{\mathbb{R}}(\gamma_p). \quad (5.15)$$

Then we obtain

$$\begin{aligned} (I - L_{(\varepsilon)}^N) P_\varepsilon f(x) &= -N \varepsilon^2 \sum_{i=1}^d \rho_{\mathbb{R}}(\gamma_p)_i \frac{\partial f}{\partial x_i}(\varepsilon \Phi_0^{(\varepsilon)}(x)) \\ &\quad - \frac{\varepsilon^2}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon \Phi_0^{(\varepsilon)}(x)) B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij}(x) + O(N^3 \varepsilon^3), \end{aligned}$$

where  $B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij} : V \rightarrow \mathbb{R}$  ( $i, j = 1, \dots, d$ ,  $N \in \mathbb{N}$ ) is defined by

$$B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij}(x) = \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_i (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_j \quad (x \in V).$$

Next we define  $\mathcal{B}_{(\varepsilon)}(\Phi_0^{(\varepsilon)})_{ij} : V_0 \rightarrow \mathbb{R}$  ( $i, j = 1, \dots, d$ ) by

$$\mathcal{B}_{(\varepsilon)}(\Phi_0^{(\varepsilon)})_{ij}(x) = \sum_{e \in (E_0)_x} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(\tilde{e})) - \Phi_0^{(\varepsilon)}(o(\tilde{e})))_i (\Phi_0^{(\varepsilon)}(t(\tilde{e})) - \Phi_0^{(\varepsilon)}(o(\tilde{e})))_j \quad (x \in V_0),$$

where  $\tilde{e}$  is a lift of  $e \in E_0$  to  $E$ . Because  $B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij} : V \rightarrow \mathbb{R}$  is  $\Gamma$ -invariant, it holds

$$\mathcal{B}_{(\varepsilon)}(\Phi_0^{(\varepsilon)})_{ij}(\pi(x)) = B_{(\varepsilon)}^1(\Phi_0^{(\varepsilon)})_{ij}(x) \quad (x \in V, i, j = 1, \dots, d).$$

Using (5.15), we obtain

$$\begin{aligned} &B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij}(x) \\ &= \sum_{c' \in \Omega_{x,N-1}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) \\ &\quad \times \left\{ (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_i + (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_i \right\} \\ &\quad \times \left\{ (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_j + (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_j \right\} \\ &= \sum_{c' \in \Omega_{x,N-1}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_i (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_j \\ &\quad + \sum_{c' \in \Omega_{x,N-1}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_i (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_j \\ &\quad + \sum_{c' \in \Omega_{x,N-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_j \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_i \\ &\quad + \sum_{c' \in \Omega_{x,N-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x))_i \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_j \\ &= L_{(\varepsilon)}^{N-1}(\mathcal{B}_{(\varepsilon)}(\Phi_0^{(\varepsilon)})_{ij})(\pi(x)) + B_{(\varepsilon)}^{N-1}(\Phi_0^{(\varepsilon)})_{ij}(x) \\ &\quad + (L_{(\varepsilon)}^{N-1}\Phi_0^{(\varepsilon)}(x) - \Phi_0^{(\varepsilon)}(x))_j \rho_{\mathbb{R}}(\gamma_{p_\varepsilon})_i + (L_{(\varepsilon)}^{N-1}\Phi_0^{(\varepsilon)}(x) - \Phi_0^{(\varepsilon)}(x))_i \rho_{\mathbb{R}}(\gamma_{p_\varepsilon})_j \end{aligned}$$



$$\begin{aligned}
&= L_{(\varepsilon)}^{N-1}(\mathcal{B}_\varepsilon(\Phi_0^{(\varepsilon)})_{ij})(\pi(x)) + B_{(\varepsilon)}^{N-1}(\Phi_0^{(\varepsilon)})_{ij}(x) + 2(N-1)\rho_{\mathbb{R}}(\gamma_{p_\varepsilon})_i \rho_{\mathbb{R}}(\gamma_{p_\varepsilon})_j \\
&= \sum_{k=0}^{N-1} L_{(\varepsilon)}^k(\mathcal{B}_\varepsilon(\Phi_0^{(\varepsilon)})_{ij})(\pi(x)) + N(N-1)\varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j.
\end{aligned}$$

By Lemma 5.1, we find that the invariant measure of  $L_{(\varepsilon)}$  is  $m$ . Then applying Proposition 3.3, we can choose a sufficiently small  $\varepsilon_0 > 0$  such that

$$\frac{1}{N} \sum_{k=0}^{N-1} L_{(\varepsilon)}^k(\mathcal{B}_\varepsilon(\Phi_0^{(\varepsilon)})_{ij})(\pi(x)) = \sum_{x \in X_0} \mathcal{B}_\varepsilon(\Phi_0^{(\varepsilon)})_{ij}(x) m(x) + O_{\varepsilon_0}\left(\frac{1}{N}\right)$$

for all  $0 \leq \varepsilon \leq \varepsilon_0$ . Furthermore, as (4.2), it follows from (3.4) that

$$\begin{aligned}
&\sum_{x \in X_0} \mathcal{B}_\varepsilon(\Phi_0^{(\varepsilon)})_{ij}(x) m(x) \\
&= \sum_{e \in E_0} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_i (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)))_j m(o(e)) \\
&= \left( \sum_{e \in E_0} p_\varepsilon(e) \omega_i^{(\varepsilon)}(e) \omega_j^{(\varepsilon)}(e) m(o(e)) - \langle \gamma_{p_\varepsilon}, \omega_i \rangle \langle \gamma_{p_\varepsilon}, \omega_j \rangle \right) + \varepsilon^2 \langle \gamma_p, \omega_i \rangle \langle \gamma_p, \omega_j \rangle \\
&= \langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)} \rangle_{(\varepsilon)} + \varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j.
\end{aligned}$$

Putting it all together, we obtain

$$\begin{aligned}
\varepsilon^2 B_{(\varepsilon)}^N(\Phi_0^{(\varepsilon)})_{ij}(x) &= N\varepsilon^2 \left( \langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)} \rangle_{(\varepsilon)} + \varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j \right) \\
&\quad + N(N-1)\varepsilon^4 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j + \varepsilon^2 O_{\varepsilon_0}(1),
\end{aligned}$$

and it implies

$$\begin{aligned}
&\frac{1}{N\varepsilon^2} (I - L_{(\varepsilon)}^N) P_\varepsilon f(x) \\
&= -\langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} f(\varepsilon \Phi_0^{(\varepsilon)}(x)) \rangle_{g_0^{(0)}} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^d \left( \langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)} \rangle_{(\varepsilon)} + \varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j \right. \\
&\quad \left. + (N-1)\varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)_i \rho_{\mathbb{R}}(\gamma_p)_j + O_{\varepsilon_0}\left(\frac{1}{N}\right) \right) \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon \Phi_0^{(\varepsilon)}(x)) + O(N^2 \varepsilon) \\
&= -\langle \rho_{\mathbb{R}}(\gamma_p), (\nabla_{(0)} f)(\varepsilon \Phi_0^{(\varepsilon)}(x)) \rangle_{g_0^{(0)}} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^d \langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)} \rangle_{(\varepsilon)} \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon \Phi_0^{(\varepsilon)}(x)) + O(N^2 \varepsilon) + O_{\varepsilon_0}\left(\frac{1}{N}\right).
\end{aligned}$$

Finally, applying Lemma 5.2, we obtain

$$\frac{1}{N\varepsilon^2} (I - L_{(\varepsilon)}^N) P_\varepsilon f(x) = P_\varepsilon \left( \frac{1}{2} \Delta_{(0)} - \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} \rangle_{g_0^{(0)}} \right) f(x) + O(N^2 \varepsilon) + O_{\varepsilon_0}\left(\frac{1}{N}\right)$$

as  $N \nearrow \infty$  and  $N^2\varepsilon \searrow 0$ . Hence we complete the proof.  $\blacksquare$

**Proof of Theorem 2.3.** Because the proof is almost same as one of Theorem 2.1, we only give a sketch. Let  $N = N(n)$  be the integer with  $n^{1/5} \leq N < 1 + n^{1/5}$ . We put  $k_N := ([nt] - [ns])/N$ ,  $\varepsilon_N := n^{-1/2}$  and  $\tau_N := N\varepsilon_N^2$ . Then  $k_N\tau_N \rightarrow (t - s)$  as  $N \rightarrow \infty$ . Then by recalling Lemma 5.4 and applying Theorem 3.5 to the case where

$$\begin{aligned} \mathcal{V} &= C_\infty((\Gamma \otimes \mathbb{R})_{(0)}), \quad \mathcal{V}_N = C_\infty(X), \quad U_N = L_{(\varepsilon_N)}^N, \\ T &= \frac{\Delta_{(0)}}{2} - \langle \rho_{\mathbb{R}}(\gamma_p), \nabla_{(0)} \rangle_{g_0^{(0)}}, \quad D = C_0^\infty((\Gamma \otimes \mathbb{R})_{(0)}), \end{aligned}$$

we complete the proof.  $\blacksquare$

### 5.3 Proof of Theorem 2.4

As is in the previous section, we complete the proof of Theorem 2.4 by showing the convergence of the finite dimensional distributions of  $\{\mathbf{Y}^{(n^{-1/2}, n)}\}_{n=1}^\infty$  (Lemma 5.5) and the tightness of  $\{\mathbf{Q}^{(n^{-1/2}, n)}\}_{n=1}^\infty$  (Lemma 5.6).

First, by recalling Lemma 5.3, we prepare

$$\begin{aligned} & \sup_{c \in \Omega_{x_*}(X)} |\mathbf{Y}_t^{(\varepsilon, n)}(c) - \mathcal{Y}_t^{(\varepsilon, n)}(c)|_{g_0^{(0)}} \\ &= \frac{(nt - [nt])}{\sqrt{n}} \sup_{c \in \Omega_{x_0}(X)} |\xi_{[nt]+1}^{(\varepsilon)}(c) - \xi_{[nt]}^{(\varepsilon)}(c)|_{g_0^{(0)}} \\ &\leq n^{-1/2} \|d\Phi_0^{(\varepsilon)}\|_\infty \\ &\leq 2 \left( \min_{e \in E_0} \tilde{m}_0(e) \right)^{-1/2} n^{-1/2} \|d\Phi_0^{(0)}\|_\infty \rightarrow 0 \end{aligned} \tag{5.16}$$

as  $n \rightarrow \infty$  uniformly for  $0 \leq \varepsilon \leq \varepsilon_0$ . We fix  $0 \leq t_1 < t_2 < \dots < t_r < \infty$  ( $r \in \mathbb{N}$ ) and set the random variable  $\mathbf{Y}_{t_1, t_2, \dots, t_r}^{(\varepsilon, n)} : \Omega_{x_*}(X) \rightarrow (\Gamma \otimes \mathbb{R})_{(0)}^r$  given by

$$\mathbf{Y}_{t_1, t_2, \dots, t_r}^{(\varepsilon, n)}(c) := \left( \mathbf{Y}_{t_1}^{(\varepsilon, n)}(c), \dots, \mathbf{Y}_{t_r}^{(\varepsilon, n)}(c) \right).$$

Noting (5.16) and Theorem 2.3, and following the proof of Lemma 4.2, we easily obtain

#### Lemma 5.5

$$\mathbf{Y}_{t_1, \dots, t_r}^{(n^{-1/2}, n)} \xrightarrow{\mathcal{D}} \left( B_{t_1} + \rho_{\mathbb{R}}(\gamma_p)t_1, \dots, B_{t_r} + \rho_{\mathbb{R}}(\gamma_p)t_r \right) \quad \text{as } n \rightarrow \infty,$$

where  $(B_t)_{t \geq 0}$  is a  $(\Gamma \otimes \mathbb{R})_{(0)}$ -valued standard Brownian motion with  $B_0 = \mathbf{0}$ .

Hence in this subsection, we concentrate on proving the following lemma:

**Lemma 5.6**  $\{\mathbf{Q}^{(n)}\}_{n=1}^\infty$  is tight in  $(\mathbf{W}_{(0)}, \mathcal{B}(\mathbf{W}_{(0)}))$ .

**Proof.** It is enough to show that there exist some  $n_0 \in \mathbb{N}$  and  $C > 0$  independent of  $n$  such that

$$\mathbb{E}^{\mathbb{P}_{x*}} \left[ \left| \mathbf{Y}_t^{(n^{-1/2}, n)} - \mathbf{Y}_s^{(n^{-1/2}, n)} \right|_{g_0^{(0)}}^4 \right] \leq C(t-s)^2 \quad (0 \leq s \leq t, n \geq n_0). \quad (5.17)$$

We set  $C_0 := 2(\min_{e \in E_0} \tilde{m}(e))^{-1/2} \|d\Phi_0^{(0)}\|_\infty$  and

$$C_1 := C_0 + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}}, \quad C_2 := C_0^4 + C_0 |\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}}^3, \quad C_3 := C_0^3 |\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}} + |\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}}^4.$$

Recalling Lemma 5.3, we have

$$\left| \sum_{e \in E_x} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) + \mathbf{x})_i^l \right| \leq (C_0 + |\mathbf{x}|_{g_0^{(0)}})^l \quad (5.18)$$

$$(x \in V, \mathbf{x} \in \Gamma \otimes \mathbb{R}, i = 1, \dots, d, l \in \mathbb{N}, 0 \leq \varepsilon \leq \varepsilon_0).$$

As in the proof of Lemma 4.3, we distinguish two cases:

$$\textbf{(I)} \quad t - s < n^{-1}, \quad \textbf{(II)} \quad t - s \geq n^{-1}.$$

First, we consider case **(I)**. By following the proof of Lemma 4.3, we easily have

$$\begin{aligned} |\mathbf{Y}_t^{(\varepsilon, n)} - \mathbf{Y}_s^{(\varepsilon, n)}|_{g_0^{(0)}} &\leq 2n^{1/2}(t-s) \|d\Phi_0^{(\varepsilon)}\|_\infty \\ &\leq 2C_0 n^{1/2}(t-s) \quad (n \in \mathbb{N}, 0 \leq \varepsilon \leq \varepsilon_0). \end{aligned}$$

This yields the desired estimate

$$\mathbb{E}^{\mathbb{P}_{x*}} \left[ \left| \mathbf{Y}_t^{(n^{-1/2}, n)} - \mathbf{Y}_s^{(n^{-1/2}, n)} \right|_{g_0^{(0)}}^4 \right] \leq 16C_0^4(t-s)^2 \quad (0 \leq s \leq t)$$

for all  $n \geq n_0 := \lceil \varepsilon_0^{-2} \rceil + 1$ , where we used  $n^2(t-s)^2 < 1$ .

Next, we consider case **(II)**. Let  $\mathcal{F}$  be the fundamental domain in  $X$  containing  $x_* \in V$  and  $M > N$  be two positive integers. As in the proof of Lemma 4.3, we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}_{x*}} \left[ \left| \mathcal{Y}_{\frac{M}{n}}^{(\varepsilon, n)} - \mathcal{Y}_{\frac{N}{n}}^{(\varepsilon, n)} \right|_{g_0^{(0)}}^4 \right] \\ &\leq d^2 n^{-2} \max_{i=1, \dots, d} \max_{x \in \mathcal{F}} \left\{ \sum_{c \in \Omega_{x, M-N}(X)} p_\varepsilon(c) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_i^4 \right\}. \end{aligned} \quad (5.19)$$

We now fix  $i = 1, \dots, d, x \in V$  and set

$$\begin{aligned} \mathfrak{M}_\varepsilon^l(k; \mathbf{x}) &:= \sum_{c \in \Omega_{x, k}(X)} p_\varepsilon(c) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x) + \mathbf{x})_i^l \\ &\quad (k = 1, \dots, M - N, l \in \mathbb{N}, \mathbf{x} \in \Gamma \otimes \mathbb{R}). \end{aligned}$$

Then we have

$$\begin{aligned}
& \mathfrak{M}_\varepsilon^4(k; j\varepsilon\rho_{\mathbb{R}}(\gamma_p)) \\
&= \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) \left\{ (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i \right. \\
&\quad \left. + (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p))_i \right\}^4 \\
&= \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^4 \\
&\quad + 4 \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p))_i \\
&\quad \times \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^3 \\
&\quad + 6 \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^2 \\
&\quad \times \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^2 \\
&\quad + 4 \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^3 \\
&\quad \times \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i \\
&\quad + \mathfrak{M}_\varepsilon^4(k-1; (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p)) \quad (k = 2, \dots, M-N, j = 0, 1, \dots). \quad (5.20)
\end{aligned}$$

Since  $\Phi_0^{(\varepsilon)}$  enjoys the modified harmonicity  $L_{(\varepsilon)}\Phi_0^{(\varepsilon)}(x) = \Phi_0^{(\varepsilon)}(x) + \varepsilon\rho_{\mathbb{R}}(\gamma_p)$  ( $x \in V$ ), the fourth term on the right-hand side of (5.20) is equal to 0. Furthermore it follows from (5.18) that

$$\sum_{e \in E_y} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^l \leq C_1^l \quad (y \in V, l = 2, 4) \quad (5.21)$$

and

$$\begin{aligned}
& \left| \sum_{c' \in \Omega_{x,k-1}(X)} p_\varepsilon(c') (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+1)\varepsilon\rho_{\mathbb{R}}(\gamma_p))_i \right. \\
&\quad \left. \times \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_{\mathbb{R}}(\gamma_p))_i^3 \right| \\
&\leq \{ (k-1)\|d\Phi_0^{(\varepsilon)}\|_\infty + (j+1)\varepsilon|\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}} \} (\|d\Phi_0^{(\varepsilon)}\|_\infty + \varepsilon|\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}})^3 \\
&\leq 4(C_0^3 + \varepsilon|\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}}^3) \{ (k-1)C_0 + (j+1)\varepsilon|\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}} \} \\
&\leq 4C_2(k-1) + 4C_3(j+1). \quad (5.22)
\end{aligned}$$

Then by combining (5.21), (5.22) with (5.20), we have

$$\begin{aligned} \mathfrak{M}_\varepsilon^4(k; j\varepsilon\rho_\mathbb{R}(\gamma_p)) &\leq C_1^4 + 4C_2(k-1) + 4C_3(j+1) + 6C_1^2\mathfrak{M}_\varepsilon^2(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &\quad + \mathfrak{M}_\varepsilon^4(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)). \end{aligned} \quad (5.23)$$

Moreover we also have

$$\begin{aligned} &\mathfrak{M}_\varepsilon^2(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &= \sum_{c' \in \Omega_{x, k-2}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) \left\{ (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_\mathbb{R}(\gamma_p))_i \right. \\ &\quad \left. + (\Phi_0^{(\varepsilon)}(t(c')) - \Phi_0^{(\varepsilon)}(x) + (j+2)\varepsilon\rho_\mathbb{R}(\gamma_p))_i \right\}^2 \\ &= \sum_{c' \in \Omega_{x, k-2}(X)} p_\varepsilon(c') \sum_{e \in E_{t(c')}} p_\varepsilon(e) (\Phi_0^{(\varepsilon)}(t(e)) - \Phi_0^{(\varepsilon)}(o(e)) - \varepsilon\rho_\mathbb{R}(\gamma_p))_i^2 \\ &\quad + \mathfrak{M}_\varepsilon^2(k-2; (j+2)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &\leq C_1^2 + \mathfrak{M}_\varepsilon^2(k-2; (j+2)\varepsilon\rho_\mathbb{R}(\gamma_p)), \end{aligned} \quad (5.24)$$

where we used the modified harmonicity again for the third line. It follows from (5.24) that

$$\begin{aligned} \mathfrak{M}_\varepsilon^2(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)) &\leq C_1^2(k-2) + \mathfrak{M}_\varepsilon^2(1; (j+k-1)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &\leq C_1^2(k-2) + (C_0 + (j+k-1)\varepsilon|\rho_\mathbb{R}(\gamma_p)|_{g_0^{(0)}})^2 \\ &\leq C_1^2(k-1) + 2C_0^2 + 2(j+k-1)^2\varepsilon^2|\rho_\mathbb{R}(\gamma_p)|_{g_0^{(0)}}^2. \end{aligned} \quad (5.25)$$

Putting (5.25) into (5.23), we obtain

$$\begin{aligned} \mathfrak{M}_\varepsilon^4(k; j\varepsilon\rho_\mathbb{R}(\gamma_p)) &\leq (2C_0^2 + C_1^4) + (6C_1^4 + C_1^2 + 4C_2)(k-1) + 4C_3(j+1) \\ &\quad + 12(j+k-1)^2\varepsilon^2|\rho_\mathbb{R}(\gamma_p)|_{g_0^{(0)}}^2 + \mathfrak{M}_\varepsilon^4(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &=: C_4 + C_5(k-1) + C_6(j+1) + C_7(j+k-1)^2\varepsilon^2 \\ &\quad + \mathfrak{M}_\varepsilon^4(k-1; (j+1)\varepsilon\rho_\mathbb{R}(\gamma_p)). \end{aligned}$$

Then we have

$$\begin{aligned} &\sum_{c \in \Omega_{x, M-N}(X)} p_\varepsilon(c) (\Phi_0^{(\varepsilon)}(t(c)) - \Phi_0^{(\varepsilon)}(x))_i^4 \\ &= \mathfrak{M}_\varepsilon^4(M-N; \mathbf{0}) \\ &\leq C_4 + C_5(M-N-1) + C_6 + C_7(M-N-1)^2\varepsilon^2 + \mathfrak{M}_\varepsilon^4(M-N-1; \varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &\leq C_4(M-N) + (C_5 + C_6) \left( \sum_{k=1}^{M-N-1} k \right) + C_7(M-N)^3\varepsilon^2 + \mathfrak{M}_\varepsilon^4(1; (M-N-1)\varepsilon\rho_\mathbb{R}(\gamma_p)) \\ &\leq (C_4 + C_5 + C_6)(M-N)^2 + C_7(M-N)^3\varepsilon^2 + (C_0 + (M-N)\varepsilon|\rho_\mathbb{R}(\gamma_p)|_{g_0^{(0)}})^4 \\ &\leq 8C_0^4 + (C_4 + C_5 + C_6)(M-N)^2 + C_7(M-N)^3\varepsilon^2 + 8|\rho_\mathbb{R}(\gamma_p)|_{g_0^{(0)}}^4(M-N)^4\varepsilon^4. \end{aligned} \quad (5.26)$$

Finally, we put  $\varepsilon = n^{-1/2}$  and  $C_8 = 8C_0^4 + C_4 + C_5 + C_6 + C_7 + 8|\rho_{\mathbb{R}}(\gamma_p)|_{g_0^{(0)}}^4$ . By (5.19) and (5.26), we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}_{x*}} \left[ \left| \mathbf{Y}_t^{(n^{-1/2}, n)} - \mathbf{Y}_s^{(n^{-1/2}, n)} \right|_{g_0^{(0)}}^4 \right] \\
& \leq \mathbb{E}^{\mathbb{P}_{x*}} \left[ \max_{i,j=0,1} \left| \mathcal{Y}_{\frac{[nt]+i}{n}}^{(n^{-1/2}, n)} - \mathcal{Y}_{\frac{[ns]+j}{n}}^{(n^{-1/2}, n)} \right|_{g_0^{(0)}}^4 \right] \\
& \leq C_8 d^2 n^{-2} \{ 1 + ([nt] - [ns] + 1)^2 + ([nt] - [ns] + 1)^3 n^{-1} + ([nt] - [ns] + 1)^4 n^{-2} \} \\
& \leq C_8 d^2 \left\{ 1 + 3 \left( t - s + \frac{2}{n} \right)^2 \right\} \\
& \leq C_8 d^2 \{ (t - s)^2 + 27(t - s)^2 \} = C(t - s)^2 \quad (n \geq n_0 = [\varepsilon_0^{-2}] + 1),
\end{aligned}$$

where we used  $[nt] - [ns] \leq n(t - s) + 1$  for the fourth line and the condition  $n^{-1} \leq (t - s)$  for the final line. Thus we obtained our desired estimate (5.17) for case **(II)**. This completes the proof.  $\blacksquare$

## 6 Asymptotic expansion of the transition probability

The main purpose of this section is to prove Theorem 2.5. Recall that  $X = (V, E)$  is a crystal lattice in which covering transformation group  $\Gamma$  is a torsion free abelian group of rank  $d$  and torsion free. Throughout this section, we always assume that the random walk  $\{w_n\}_{n=1}^\infty$  on  $X$  is *irreducible*, and let  $K$  be the period of the random walk on  $X$ . As mentioned in Section 2, this assumption implies the irreducibility of the corresponding random walk  $\{\pi(w_n)\}_{n=0}^\infty$  on  $X_0 = (V_0, E_0)$ , and let  $K_0$  be the period of the random walk on  $X_0$ . However it should be remarked that  $K$  does not coincide with  $K_0$  in general. To overcome this difficulty, the following lemma plays a key role.

**Lemma 6.1 (cf. [15])** *Suppose that the random walk on the crystal lattice  $X$  is of period  $K$ . Then there exists a subgroup  $\Gamma_1$  of  $\Gamma$  with  $\text{rank}(\Gamma_1) = d$  such that the corresponding random walk on the quotient graph  $X_1 = \Gamma_1 \backslash X$  has the same period  $K$ .*

**Proof.** Let  $V = \coprod_{k=0}^{K-1} A_k$  be the corresponding  $K$ -partition of  $V$ . For any  $x_0 \in A_0$ , we set

$$\Gamma_1 := \{ \sigma \in \Gamma \mid \sigma x_0 \in A_0 \}.$$

It is easy to see that  $\Gamma_1$  is a subgroup of  $\Gamma$  with rank  $d$ . Thus it suffices to show that  $\Gamma_1$  preserves the  $K$ -partition. For arbitrary  $x_k \in A_k$  and  $\sigma \in \Gamma_1$ , the step of the walk from  $\sigma x_0 \in A_0$  to  $\sigma x_k$  is same as one from  $x_0$  to  $x_k$ , which concludes that  $\sigma x_k \in A_k$ .  $\blacksquare$

Henceforth, without loss of generality, we may assume that the covering transformation group  $\Gamma$  preserves the  $K$ -partition, which implies that the corresponding random walk on the quotient graph  $X_0 = \Gamma \backslash X$  has the same period  $K$ .

**Example 6.2** Let  $X = (V, E)$  be the 2-dimensional square lattice with the covering transformation group  $\Gamma = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2$ . Its quotient graph is a 2-bouquet graph  $X_0$  (see Figure 1). Then the period of the simple random walk on  $X$  and the one of the corresponding random walk on  $X_0$  are 2 and 1, respectively. On the other hand, if we replace  $\Gamma$  by a subgroup  $\Gamma_1 = \langle \sigma'_1, \sigma'_2 \rangle$  defined as in Figure 2, the period of the corresponding random walk on  $X_1$  is equal to 2.

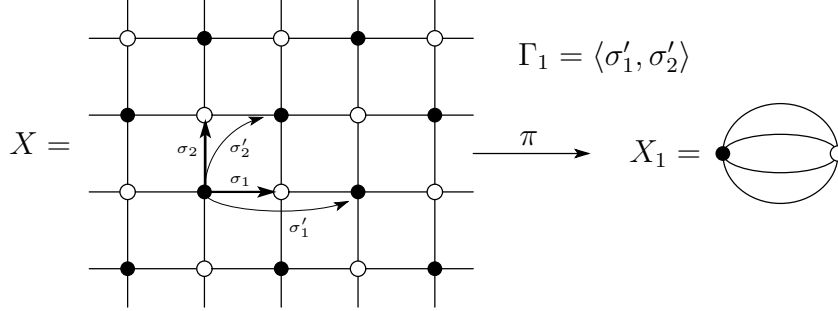


Figure 2: Square lattice graph and the quotient preserving the bipartition

## 6.1 Preliminaries from the twisted transition operators

For the reader's convenience, we first review some basic results on the twisted transition operators studied in [10, 11, 15]. Let  $\widehat{H}_1(X_0, \mathbb{Z})$  be the group of unitary characters of  $H_1(X_0, \mathbb{Z})$ . We identify  $\widehat{H}_1(X_0, \mathbb{Z})$  with the Jacobian torus

$$J(X_0) := H^1(X_0, \mathbb{R})/H^1(X_0, \mathbb{Z})$$

by the mapping

$$H^1(X_0, \mathbb{R}) \ni \omega \mapsto \chi_\omega \in \widehat{H}_1(X_0, \mathbb{Z}),$$

where

$$\chi_\omega(\sigma) := \exp\left(2\pi\sqrt{-1} \int_{c_\sigma} \omega\right) \quad (\sigma \in \Gamma)$$

and  $c_\sigma$  is a closed path in  $X_0$  satisfying  $\rho([c_\sigma]) = \sigma$ . We equip a flat metric on  $J(X_0)$  induced by the metric (3.3) on  $H^1(X_0, \mathbb{R}) (\cong \mathcal{H}^1(X_0))$ .

Let  $\widehat{\Gamma}$  be the group of unitary characters of the covering transformation group  $\Gamma$ . By the above mapping, we can also identify  $\widehat{\Gamma}$  with the  $\Gamma$ -Jacobian torus

$$\text{Jac}^\Gamma := \text{Hom}(\Gamma, \mathbb{R})/\text{Hom}(\Gamma, \mathbb{Z}).$$

The canonical surjective homomorphism  $\rho : H_1(X_0, \mathbb{Z}) \rightarrow \Gamma$  gives rise to an injective homomorphism  $\text{Jac}^\Gamma$  into  $J(X_0)$ . We regard  $\text{Jac}^\Gamma$  as the flat torus with the metric induced by that on  $J(X_0)$ . The tangent space  $T_1 \widehat{\Gamma}$  at the trivial character  $\mathbf{1}$  coincides with  $\{\omega \in$

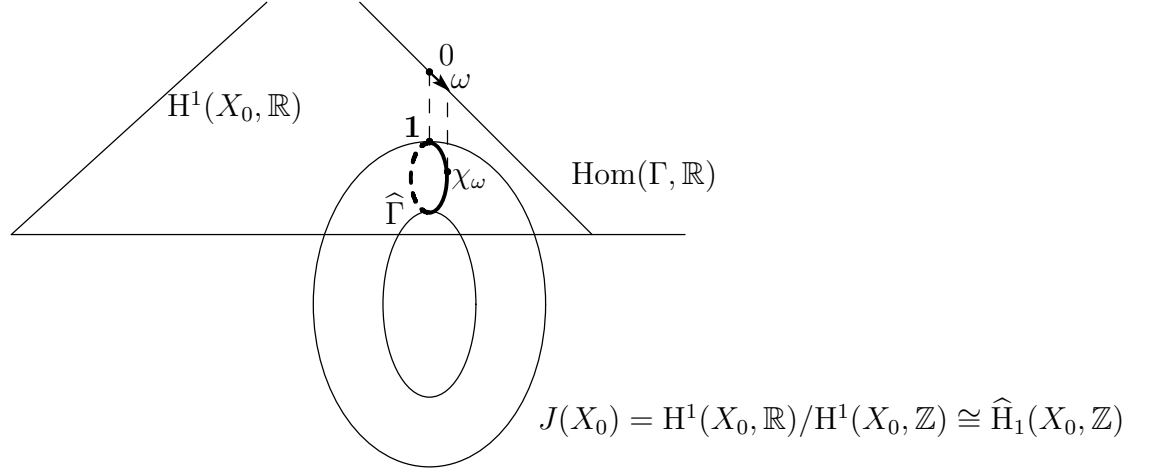


Figure 3:  $\widehat{\Gamma} \subset J(X_0)$  and  $\text{Hom}(\Gamma, \mathbb{R}) \subset H^1(X_0, \mathbb{R})$ .

$H^1(X_0, \mathbb{R})|_{\chi_\omega \in \widehat{\Gamma}}$ , and it is identified with  $\text{Hom}(\Gamma, \mathbb{R})$  (see Figure 3). Since the lattice group  $\Gamma \otimes \mathbb{Z}$  in  $\Gamma \otimes \mathbb{R}$  and the lattice group  $\text{Hom}(\Gamma, \mathbb{Z})$  in  $\text{Hom}(\Gamma, \mathbb{R})$  are dual each other, we observe that the  $\Gamma$ -Albanese torus  $\text{Alb}^\Gamma = (\Gamma \otimes \mathbb{R} / \Gamma \otimes \mathbb{Z}, g_0)$  is the dual flat torus of  $\text{Jac}^\Gamma$ , and hence  $\text{vol}(\widehat{\Gamma}) = \text{vol}(\text{Jac}^\Gamma) = \text{vol}(\text{Alb}^\Gamma)^{-1}$ .

To analyze the  $n$ -step transition probability  $p(n, x, y)$  for the random walk on the crystal lattice  $X = (V, E)$ , we need to introduce the *twisted transition operator*  $L_\chi$  for a unitary character  $\chi \in \widehat{\Gamma}$ . For each  $\chi \in \widehat{\Gamma}$ , we consider the  $|V_0|$ -dimensional inner product space

$$\ell_\chi^2 = \{f : X \rightarrow \mathbb{C} \mid f(\sigma x) = \chi(\sigma)f(x) \text{ for } \sigma \in \Gamma\}$$

with the inner product

$$\langle f, g \rangle_\chi = \sum_{x \in \mathcal{F}} f(x) \overline{g(x)},$$

where  $\mathcal{F} \subset V$  is a fundamental domain of  $X$  for  $\Gamma$ . We note that the inner product is independent of the choice of a fundamental domain  $\mathcal{F}$ .

As the transition operator  $L$  and its transpose  ${}^tL$  preserve  $\ell_\chi^2$  (see [15]), we define the *twisted transition operator*  $L_\chi : \ell_\chi^2 \rightarrow \ell_\chi^2$  and its transposed operator  ${}^tL_\chi : \ell_\chi^2 \rightarrow \ell_\chi^2$  by the restriction of  $L$  and  ${}^tL$ , respectively. For the trivial character  $\chi = \mathbf{1}$ ,  $(L_1, \ell_1^2)$  and  $({}^tL_1, \ell_1^2)$  are identified with  $(L, \ell^2(X_0))$  and  $({}^tL, \ell^2(X_0))$ , respectively. The family  $\{L_\chi\}_{\chi \in \widehat{\Gamma}}$  gives rise to the direct integral decomposition

$$(L, \ell^2(X)) = \int_{\widehat{\Gamma}}^{\oplus} (L_\chi, \ell_\chi^2) d\chi,$$

where  $d\chi$  denotes the normalized Haar measure on  $\widehat{\Gamma}$ . As in [15, Section 7], this decom-



position implies an integral expression of the  $n$ -step transition probability

$$p(n, x, y) = \int_{\widehat{\Gamma}} \langle L_{\chi}^n f_y, f_x \rangle_{\chi} d\chi \quad (n \in \mathbb{N}, x, y \in V), \quad (6.1)$$

where  $f_x \in \ell_{\chi}^2$  is the modified delta function defined by

$$f_x(z) := \begin{cases} \chi(\sigma) & \text{if } z = \sigma x, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from  $\langle L_{\chi}^n f_y, f_x \rangle_{\chi} = 0$  ( $x \in A_i, y \in A_j, n \neq Kl + j - i$ ) that  $p(n, x, y) = 0$ .

By virtue of the Perron–Frobenius theorem for the random walk with period  $K$ , the twisted transition operator  $L_{\chi}$  has  $K$  maximal eigenvalues  $\mu_0(\chi), \dots, \mu_{K-1}(\chi)$ . For the eigenvalue  $\mu_k(\chi)$  ( $k = 0, \dots, K-1$ ), we denote the corresponding right eigenfunction and left eigenfunction by  $\phi_{k,\chi}$  and  $\psi_{k,\chi}$ , respectively. Specifically,

$$L_{\chi} \phi_{k,\chi} = \mu_k(\chi) \phi_{k,\chi}, \quad {}^t L_{\chi} \psi_{k,\chi} = \overline{\mu_k(\chi)} \psi_{k,\chi}.$$

Normalizing  $\phi_{k,\chi}$  and  $\psi_{k,\chi}$ , we may assume

$$\langle \phi_{k,\chi}, \phi_{k,\chi} \rangle_{\chi} = \langle \phi_{k,\chi}, \psi_{k,\chi} \rangle_{\chi} = 1.$$

It should be noted that the function  $\mu_k = \mu_k(\chi)$  is a continuous function in  $\chi \in \widehat{\Gamma}$  and

$$\mu_k(\mathbf{1}) = \exp\left(\frac{2\pi k}{K} \sqrt{-1}\right) \quad (k = 0, \dots, K-1).$$

Applying Proposition 3.4, we observe that the eigenvalue  $\mu_k(\chi)$  is simple and  $\mu_k(\chi)$ ,  $\phi_{k,\chi}$ ,  $\psi_{k,\chi}$  are smooth in a small neighborhood  $U(\mathbf{1})(\subset \widehat{\Gamma})$  of the trivial character  $\mathbf{1}$ . Moreover, the operator norm  $\|L_{\chi}\|$  is equal to 1 if and only if  $\chi = \mathbf{1}$ . (see [10, 15] for details.)

We next decompose  $\ell_{\chi}^2$  as  $\ell_{\chi}^2 = \oplus_{k=0}^{K-1} \langle \phi_{k,\chi} \rangle \oplus \mathcal{V}_{\chi}$ , where

$$\mathcal{V}_{\chi} := \{f \in \ell_{\chi}^2 \mid \langle f, \psi_{k,\chi} \rangle_{\chi} = 0 \ (0 \leq k \leq K-1)\}.$$

More precisely, for  $f \in \ell_{\chi}^2$ , there exists a unique  $f_{\mathcal{V}_{\chi}} \in \mathcal{V}_{\chi}$  such that

$$f = \sum_{k=0}^{K-1} \langle f, \psi_{k,\chi} \rangle_{\chi} \phi_{k,\chi} + f_{\mathcal{V}_{\chi}}. \quad (6.2)$$

Combining (6.1) with (6.2), we obtain

$$p(n, x, y) = \int_{\widehat{\Gamma}} \left( \sum_{k=0}^{K-1} \mu_k(\chi)^n \phi_{k,\chi}(x) \overline{\psi_{k,\chi}(y)} + \langle L_{\chi}^n \{(f_y)_{\mathcal{V}_{\chi}}\}, f_x \rangle_{\chi} \right) d\chi \quad (6.3)$$

for  $x \in A_i$ ,  $y \in A_j$  and  $n = Kl + j - i$ . Since  $L_\chi$  preserves  $\mathcal{V}_\chi$ , and  $\|L_\chi|_{\mathcal{V}_\chi}\| < 1 - \varepsilon$  for some  $\varepsilon > 0$  uniformly in  $\chi \in \widehat{\Gamma}$  (see [10, 18]), we have

$$\left| \int_{\widehat{\Gamma}} \langle L_\chi^n \{ (f_y)_{\mathcal{V}_\chi} \}, f_x \rangle_\chi d\chi \right| \leq C(1 - \varepsilon)^n \quad (6.4)$$

for some positive constant  $C$  independent of  $x$  and  $y$ . Therefore it suffices to discuss an precise asymptotic behavior of the first term of (6.3) for proving Theorem 2.5.

To look more closely at the first term of the integrand in (6.3), we make use of the correspondence  $\omega \mapsto \chi_\omega = \exp(2\pi\sqrt{-1}\langle\omega, \cdot\rangle)$  between small  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$  and  $\chi_\omega \in U(\mathbf{1})$ . For each  $\omega \in \text{Hom}(\Gamma, \mathbb{R}) \subset H^1(X_0, \mathbb{R})$ , we define a function  $s_\omega \in \ell^2_{\chi_\omega}$  by

$$s_\omega(x) := \exp\left(2\pi\sqrt{-1} \int_{x_*}^x \tilde{\omega}\right) = \exp\left(2\pi\sqrt{-1}\langle\omega, \Phi_0(x)\rangle\right) \quad (x \in V),$$

where  $x_* \in V$  is a fixed base point with  $\Phi_0(x_*) = \mathbf{0}$  and  $\tilde{\omega}$  stands for the lift of  $\omega$  to  $X$ . We define a unitary map  $S : \ell^2(X_0) \rightarrow \ell^2_{\chi_\omega}$  by

$$(Sf)(x) := \tilde{f}(x)s_\omega(x) \quad (x \in V),$$

where  $\tilde{f}$  is the lift of  $f \in \ell^2(X_0)$  to  $X$ .

We now introduce operators  $H_\omega := S^{-1}L_{\chi_\omega}S$  and  $H_\omega^* := S^{-1}({}^tL_{\chi_\omega})S$  acting on  $\ell^2(X_0)$ . Then we have

$$\begin{aligned} H_\omega f(x_0) &:= \sum_{e \in (E_0)_{x_0}} p(e) \exp(2\pi\sqrt{-1}\omega(e)) f(t(e)) \quad (x_0 \in V_0), \\ H_\omega^* f(x_0) &:= \sum_{e \in (E_0)_{x_0}} p(\bar{e}) \exp(2\pi\sqrt{-1}\omega(e)) f(t(e)) \quad (x_0 \in V_0). \end{aligned}$$

We put  $\phi_{k,\omega} := S^{-1}\phi_{k,\chi_\omega}$  and  $\psi_{k,\omega} := S^{-1}\psi_{k,\chi_\omega}$  for  $k = 0, \dots, K-1$ . Then we easily see that  $\phi_{k,\omega}$  and  $\psi_{k,\omega}$  are the eigenfunctions of the operators  $H_\omega$  and  $H_\omega^*$  corresponding to the eigenvalues  $\mu_k(\chi_\omega)$  and  $\overline{\mu_k(\chi_\omega)}$ , respectively. It is obvious that  $\mu_k(\chi_\omega)$ ,  $\phi_{k,\omega}$  and  $\psi_{k,\omega}$  depend smoothly on  $\omega$  around 0. Applying the Perron–Frobenius theorem and Proposition 3.4, we easily see

$$\mu_k(\chi_\omega) = \exp\left(\frac{2k\pi\sqrt{-1}}{K}\right) \mu_0(\chi_\omega), \quad \phi_{k,\omega} = T^k \phi_{0,\omega}, \quad \psi_{k,\omega} = T^k \psi_{0,\omega}$$

for sufficiently small  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ , where  $T : \ell^2(X_0) \rightarrow \ell^2(X_0)$  is defined by

$$Tf(x_0) = \exp\left(\frac{2k\pi\sqrt{-1}}{K}\right) f(x_0) \quad (x_0 \in \pi(A_k), \ k = 0, \dots, K-1).$$

Then we obtain

$$\begin{aligned} &\mu_k(\chi_\omega)^n \phi_{k,\chi_\omega}(x) \overline{\psi_{k,\chi_\omega}(y)} \\ &= \exp\left(\frac{2k(n+i-j)\pi\sqrt{-1}}{K}\right) \mu_0(\chi_\omega)^n \phi_{0,\omega}(\pi(x)) \overline{\psi_{0,\omega}(\pi(y))} \exp\left(2\pi\sqrt{-1} \int_y^x \tilde{\omega}\right) \\ &= \mu_0(\chi_\omega)^n \phi_{0,\omega}(\pi(x)) \overline{\psi_{0,\omega}(\pi(y))} \exp\left(-2\pi\sqrt{-1}\langle\omega, \Phi_0(y) - \Phi_0(x)\rangle\right) \end{aligned} \quad (6.5)$$

for  $x \in A_i, y \in A_j, n = Kl + j - i, k = 0, \dots, K - 1$  and sufficiently small  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ .

Now, we are going to analyze the behavior of  $\mu_0(\chi_\omega)$ ,  $\phi_{0,\chi_\omega}$  and  $\psi_{0,\chi_\omega}$  around  $\omega = 0$ . For this sake, we take a path  $\chi_t = \chi_{t\omega}$  in  $\widehat{\Gamma}$  through the trivial character  $\mathbf{1}$  at  $t = 0$ . We put

$$H_t = H_{\chi_{t\omega}}, \quad H_t^* = H_{\chi_{t\omega}}^*, \quad \phi_t = \phi_{t\omega} = \phi_{0,\chi_{t\omega}}, \quad \psi_t = \psi_{t\omega} = \psi_{0,\chi_{t\omega}}, \quad \lambda_\omega(t) = -\log \mu_0(\chi_{t\omega})$$

near  $t = 0$ . In the sequel, we denote the  $k$ -th derivative of functions  $\lambda_\omega(t)$ ,  $\phi_t(x_0)$  and  $\psi_t(x_0)$  in  $t$  by  $\lambda_\omega^{(k)}(t)$ ,  $\phi_t^{(k)}(x_0)$  and  $\psi_t^{(k)}(x_0)$ , respectively.

To prove Theorem 2.5, we need  $\lambda^{(i)}(0)$  ( $0 \leq i \leq 4$ ) and  $\phi_0^{(j)}(x_0), \psi_0^{(j)}(y_0)$  ( $0 \leq j \leq 2$ ). Differentiating both sides of  $H_t \phi_t = \exp(-\lambda_\omega(t)) \phi_t$  in  $t$  at  $t = 0$ , and following the argument in [11, Lemma 3.2], we obtain

**Lemma 6.3**

$$\begin{aligned} \lambda(0) &= 0, \\ \lambda'(0) &= -2\pi\sqrt{-1}\langle\gamma_p, \omega\rangle, \\ \lambda''(0) &= 4\pi^2\left(\sum_{e \in E_0} p(e)\omega(e)^2 m(o(e)) - \langle\gamma_p, \omega\rangle^2\right) = 4\pi^2\|\omega\|^2, \\ \lambda^{(3)}(0) &= 8\pi^3\sqrt{-1}\sum_{e \in E_0} p(e)\omega(e)^3 m(o(e)) - 24\pi^2\sqrt{-1}\|\omega\|^2\langle\gamma_p, \omega\rangle - 8\pi^3\sqrt{-1}\langle\gamma_p, \omega\rangle^3 \\ &\quad - 6\pi\sqrt{-1}|V_0|^{1/2}\sum_{e \in E_0} p(e)\omega(e)d\phi_0''(e)m(o(e)), \\ \lambda^{(4)}(0) &= -16\pi^4\sum_{e \in E_0} p(e)\omega(e)^4 m(o(e)) + 48\pi^4\|\omega\|^4 \\ &\quad + 64\pi^4\langle\gamma_p, \omega\rangle\sum_{e \in E_0} p(e)\omega(e)^3 m(o(e)) - 96\pi^4\langle\gamma_p, \omega\rangle^2\|\omega\|^2 - 48\pi^4\langle\gamma_p, \omega\rangle^4 \\ &\quad - 48\pi^2|V_0|^{1/2}\langle\gamma_p, \omega\rangle\sum_{e \in E_0} p(e)\omega(e)d\phi_0''(e)m(o(e)) \\ &\quad + 24\pi^2|V_0|^{1/2}\sum_{e \in E_0} p(e)\omega(e)^2\left(\phi_0''(t(e)) - \sum_{z \in V_0} \phi_0''(z)m(z)\right)m(o(e)) \\ &\quad - 8\pi\sqrt{-1}|V_0|^{1/2}\sum_{e \in E_0} p(e)\omega(e)d\phi_0^{(3)}(e)m(o(e)). \end{aligned}$$

**Remark 6.4** Differentiating both sides of  $H_t^* \psi_t = \exp(-\overline{\lambda_\omega(t)}) \psi_t$  four times in  $t$  at  $t = 0$ , we also obtain

$$\begin{aligned} \lambda^{(3)}(0) &= 8\pi^3\sqrt{-1}\sum_{e \in E_0} p(e)\omega(e)^3 m(o(e)) - 24\pi^2\sqrt{-1}\|\omega\|^2\langle\gamma_p, \omega\rangle - 8\pi^3\sqrt{-1}\langle\gamma_p, \omega\rangle^3 \\ &\quad + 12\pi^2|V_0|^{-1/2}\sum_{e \in E_0} p(e)\omega(e)^2\overline{\psi_0'(o(e))}, \end{aligned}$$

and

$$\begin{aligned}
\lambda^{(4)}(0) = & -16\pi^4 \sum_{e \in E_0} p(e)\omega(e)^4 m(o(e)) + 48\pi^4 \|\omega\|^4 \\
& + 64\pi^4 \langle \gamma_p, \omega \rangle \sum_{e \in E_0} p(e)\omega(e)^3 m(o(e)) - 96\pi^4 \langle \gamma_p, \omega \rangle^2 \|\omega\|^2 \\
& - 48\pi^4 \langle \gamma_p, \omega \rangle^4 - 96\pi^3 \sqrt{-1} |V_0|^{-1/2} \langle \gamma_p, \omega \rangle \sum_{e \in E_0} p(e)\omega(e)^2 \overline{\psi'_0(o(e))} \\
& + 32\pi^3 \sqrt{-1} |V_0|^{-1/2} \sum_{e \in E_0} p(e)\omega(e)^3 \overline{\psi'_0(o(e))} \\
& + 24\pi^2 |V_0|^{-1/2} \sum_{e \in E_0} p(e)\omega(e)^2 \left( \psi''_0(o(e)) - \sum_{z \in V_0} \psi''_0(z) \cdot m(o(e)) \right).
\end{aligned}$$

Changing eigensections  $s_\omega$  as in the proof of Lemma 6.6 below, if necessary, we obtain the following:

**Lemma 6.5**

$$\phi_0(x_0) = |V_0|^{-1/2}, \quad \psi_0(x_0) = |V_0|^{1/2} m(x_0), \quad \phi'_0(x_0) = 0 \quad (x_0 \in V_0).$$

Furthermore  $\psi'_0$  is a purely imaginary-valued first order polynomial of  $\omega$  satisfying

$$\begin{cases} (I - {}^t L) \psi'_0(x_0) = 2\pi \sqrt{-1} |V_0|^{1/2} \left( \sum_{e \in (E_0)_{x_0}} p(\bar{e}) \omega(e) m(t(e)) \right. \\ \quad \left. - m(x_0) \sum_{e \in E_0} p(\bar{e}) \omega(e) m(t(e)) \right) & (x_0 \in V_0) \\ \sum_{z \in V_0} \psi'_0(z) = 0, \end{cases}$$

and  $\phi''_0$  and  $\psi''_0$  are real-valued second order polynomials of  $\omega$ , satisfying

$$\begin{cases} (I - L) \phi''_0(x_0) = -4\pi^2 |V_0|^{-1/2} \left( \sum_{e \in (E_0)_{x_0}} p(e) \omega(e)^2 \right. \\ \quad \left. - \sum_{e \in E_0} p(e) \omega(e)^2 m(o(e)) \right) & (x_0 \in V_0) \\ \sum_{z \in V_0} \phi''_0(z) = 0 \end{cases}$$

and

$$\begin{cases} (I - {}^t L) \psi''_0(x_0) = 4\pi \sqrt{-1} \left( \sum_{e \in (E_0)_{x_0}} p(\bar{e}) \omega(e) \psi'_0(t(e)) + \langle \gamma_p, \omega \rangle \psi'_0(x_0) \right) \\ \quad - 4\pi^2 \left( \sum_{e \in (E_0)_{x_0}} p(\bar{e}) \omega(e)^2 \psi_0(t(e)) \right. \\ \quad \left. - m(x_0) \sum_{e \in E_0} p(\bar{e}) \omega(e)^2 \psi_0(t(e)) \right) & (x_0 \in V_0) \\ \sum_{z \in V_0} \psi''_0(z) = -|V_0| \sum_{z \in V_0} \phi''_0(z) m(z), \end{cases}$$

respectively.

The following lemma plays a key role in the proof of Theorem 2.5.

**Lemma 6.6** (cf. [11, Lemma 3.2]) *For any  $k \in \mathbb{N}$ , by changing eigensections  $s_\omega$  if necessary,  $\lambda_\omega^{(i)}(0)$ ,  $\phi_0^{(i)}(x_0)$ ,  $\psi_0^{(i)}(x_0)$  ( $1 \leq i \leq k$ ,  $x_0 \in V_0$ ) are the  $i$ -th order real coefficient homogeneous polynomials of  $\sqrt{-1}\omega$ . Here the polynomial of  $\sqrt{-1}\omega$  means that the polynomial of  $\sqrt{-1}u_1, \dots, \sqrt{-1}u_d$  when  $\omega = u_1\omega_1 + u_2\omega_2 + \dots + u_d\omega_d$ , where  $\{\omega_1, \dots, \omega_d\}$  is an orthonormal basis of  $\text{Hom}(\Gamma, \mathbb{R})$ .*

**Proof.** We proceed by induction on  $k$ . By Lemmas 6.3 and 6.5, we easily see that the desired assertion holds for  $k = 1$ . Assume that the assertion is true for all  $i \leq k - 1$ . We recall that the eigenvalue  $\lambda_\omega(t)$  and the corresponding eigenfunctions  $\phi_t, \psi_t$  satisfy

$$H_t \phi_t(x_0) = e^{-\lambda(t)} \phi_t(x_0), \quad H_t^* \psi_t(x_0) = e^{-\overline{\lambda(t)}} \psi_t(x_0) \quad (x_0 \in V_0).$$

Taking the  $k$ -th derivative of both sides and letting  $t \rightarrow 0$ , we have

$$(I - L)\phi_0^{(k)}(x_0) = \sum_{i=0}^{k-1} \binom{k}{i} \left\{ \sum_{e \in (E_0)_{x_0}} p(e) (2\pi\sqrt{-1}\omega(e))^{k-i} \phi_0^{(i)}(t(e)) - \left(\frac{d}{dt}\right)^{k-i} \Big|_{t=0} e^{-\lambda(t)} \cdot \phi_0^{(i)}(x_0) \right\} \quad (x_0 \in V_0), \quad (6.6)$$

$$(I - {}^tL)\psi_0^{(k)}(x_0) = \sum_{i=0}^{k-1} \binom{k}{i} \left\{ \sum_{e \in (E_0)_{x_0}} p(\overline{e}) (2\pi\sqrt{-1}\omega(e))^{k-i} \psi_0^{(i)}(t(e)) - \left(\frac{d}{dt}\right)^{k-i} \Big|_{t=0} e^{-\overline{\lambda(t)}} \cdot \psi_0^{(i)}(x_0) \right\} \quad (x \in V_0). \quad (6.7)$$

Taking sum of both sides of (6.6) for  $x_0 \in V_0$  with the invariant measure  $m$ , the left-hand side vanishes. By the assumption, we find that  $\lambda^{(k)}(0)$  is a desired polynomial. We can obtain the same result by taking sum of the both sides of (6.7) for  $x_0 \in V_0$ .

In the case  $k$  is even, substituting  $\lambda^{(k)}(0)$  into (6.6) and (6.7), we observe that

$$(I - L) \text{Im} \phi_0^{(k)} = 0, \quad (I - {}^tL) \text{Im} \psi_0^{(k)} = 0.$$

These imply that  $\text{Im} \phi_0^{(k)} = a_k(\omega)$  and  $\text{Im} \psi_0^{(k)} = b_k(\omega)m$  for some constant functions  $a_k(\omega)$  and  $b_k(\omega)$  on  $V_0$  depending on  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ . Clearly,  $a_k(s\omega) = s^k a_k(\omega)$  and  $b_k(s\omega) = s^k b_k(\omega)$  hold. Here we note that

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)^k \Big|_{t=0} \langle \phi_t, \psi_t \rangle_{\ell^2(X_0)} \\ &= |V_0|^{1/2} \sum_{z \in V_0} \phi_0^{(k)}(z) m(z) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{z \in V_0} \phi_0^{(k-i)}(z) \overline{\psi_0^{(i)}(z)} + |V_0|^{-1/2} \sum_{z \in V_0} \overline{\psi_0^{(k)}(z)}. \end{aligned} \quad (6.8)$$

Taking the imaginary part, by the assumption of  $\phi_0^{(i)}$  and  $\psi_0^{(i)}$  for  $i \leq k$ , we obtain  $b_k(\omega) = |V_0| a_k(\omega)$ . By replacing eigensection  $s_\omega$  with  $s_\omega \exp\left(\sqrt{-1} \frac{a_k(\omega)}{k!} |V_0|^{1/2}\right)$ , and eigenfunctions

$\phi_\omega, \psi_\omega$  with  $\phi_\omega \exp\left(-\sqrt{-1}\frac{a_k(\omega)}{k!}|V_0|^{1/2}\right), \psi_\omega \exp\left(-\sqrt{-1}\frac{a_k(\omega)}{k!}|V_0|^{1/2}\right)$ , respectively, we can assume that  $a_k = b_k = 0$ . Then  $\phi_0^{(k)}$  and  $\psi_0^{(k)}$  are real functions satisfying (6.6), (6.7), respectively. Since the right-hand side of (6.6) and (6.7) are  $k$ -th order homogeneous polynomials of  $\omega$ :

$$\sum_{|\alpha|=k} \mathcal{C}_\alpha^1(x_0)u^\alpha, \quad \sum_{|\alpha|=k} \mathcal{C}_\alpha^2(x_0)u^\alpha,$$

where  $u^\alpha = u_1^{\alpha_1} \cdots u_d^{\alpha_d}$  and  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$  is a multi-index.  $\phi_0^{(k)}$  and  $\psi_0^{(k)}$  are also  $k$ -th order homogeneous polynomials of  $\omega$  up to  $\text{Ker}(I - L)$  and  $\text{Ker}(I - {}^tL)$ , respectively. Namely, for  $\omega = u_1\omega_1 + \cdots + u_d\omega_d$ ,

$$\phi_0^{(k)}(x_0) = \sum_{|\alpha|=k} \phi_\alpha(x_0)u^\alpha + c_k(\omega), \quad \psi_0^{(k)}(x_0) = \sum_{|\alpha|=k} \psi_\alpha(x_0)u^\alpha + d_k(\omega)m(x_0),$$

where  $\phi_\alpha$  and  $\psi_\alpha$  are respectively solutions of

$$(I - L)\phi_\alpha(x_0) = \mathcal{C}_\alpha^1(x_0), \quad (I - {}^tL)\psi_\alpha(x_0) = \mathcal{C}_\alpha^2(x_0) \quad (x \in V_0).$$

By the assumption of  $\langle \phi_t, \phi_t \rangle_{\ell^2(X_0)} = 1$ , we have

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right)^k \Big|_{t=0} \langle \phi_t, \phi_t \rangle_{\ell^2(X_0)} \\ &= |V_0|^{-1/2} \sum_{z \in V_0} \phi_0^{(k)}(z) + \sum_{i=1}^{k-1} \binom{k}{i} \sum_{z \in V_0} \phi_0^{(k-i)}(z) \overline{\phi_0^{(i)}(z)} + |V_0|^{-1/2} \sum_{z \in V_0} \overline{\phi_0^{(k)}(z)}. \end{aligned} \quad (6.9)$$

Together with (6.8), we find that  $c_k(\omega)$  and  $d_k(\omega)$  are  $k$ -th order homogeneous polynomials of  $\sqrt{-1}\omega$ . Hence, we conclude that  $\phi_0^{(k)}(x_0)$  and  $\psi_0^{(k)}(x_0)$  are  $k$ -th order homogeneous polynomial of  $\sqrt{-1}\omega$ .

In the case  $k$  is odd, we see that

$$(I - L) \text{Re } \phi_0^{(k)} = 0, \quad (I - {}^tL) \text{Re } \psi_0^{(k)} = 0,$$

which implies that  $\text{Re } \phi_0^{(k)} = a_k(\omega)$ ,  $\text{Re } \psi_0^{(k)} = b_k(\omega)m$  for some constant function  $a_k, b_k$  on  $V_0$  depending on  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ . As before, we see that  $a_k(s\omega) = s^k a_k(\omega)$  and  $b_k(s\omega) = s^k b_k(\omega)$ . Taking the real part of  $(d/dt)^k|_{t=0} \langle \phi_t, \psi_t \rangle_{\ell^2(X_0)}$  as above, we obtain  $a_k = |V_0|b_k$ . Additionally, using (6.9), we have

$$2|V_0|^{-1/2} \sum_{z \in V_0} \text{Re } \phi_0^{(k)}(z) = 0,$$

which implies that  $a_k = b_k = 0$ . Then we see that  $\phi_0^{(k)}$  and  $\psi_0^{(k)}$  are imaginary-valued functions on  $V_0$  written as

$$\begin{aligned} \phi_0^{(k)}(x_0) &= \sum_{|\alpha|=k} \sqrt{-1}a_\alpha \phi_\alpha(x_0) + \sqrt{-1}c_k(\omega), \\ \psi_0^{(k)}(x_0) &= \sum_{|\alpha|=k} \sqrt{-1}b_\alpha \psi_\alpha(x_0) + \sqrt{-1}d_k(\omega)m(x_0) \end{aligned}$$

for some constant functions  $c_k(\omega), d_k(\omega)$  on  $V_0$  depending on  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$ . As before, it is easy to see that  $c_k(s\omega) = s^k c_k(\omega)$ ,  $d_k(s\omega) = s^k d_k(\omega)$ . Then, by replacing eigensection  $s_\omega$  with  $s_\omega \exp\left(\sqrt{-1} \frac{c_k(\omega)}{k!} |V_0|^{1/2}\right)$ , and eigenfunctions  $\phi_\omega, \psi_\omega$  with  $\phi_\omega \exp\left(-\sqrt{-1} \frac{c_k(\omega)}{k!} |V_0|^{1/2}\right)$ ,  $\psi_\omega \exp\left(-\sqrt{-1} \frac{c_k(\omega)}{k!} |V_0|^{1/2}\right)$ , respectively, we can assume that  $c_k = 0$ , which implies that  $\phi_0^{(k)}(x_0)$  is a  $k$ -th order homogeneous polynomial of  $\sqrt{-1}\omega$ .

Finally, from (6.8) and the assumption of the induction, we obtain that  $d_k$  is a  $k$ -th order homogeneous polynomial, which completes the proof.  $\blacksquare$

## 6.2 Basic facts on the Fourier transform

In this subsection, we quickly review several basic facts on the Fourier transform based on [11, Section 4]. We define the function  $\mathfrak{F}f(\xi) = \mathfrak{F}_{2\pi^2}f(\xi)$ , for a rapidly decreasing function  $f = f(u)$  ( $u \in \mathbb{R}^d$ ), by

$$\mathfrak{F}f(\xi) := \int_{\mathbb{R}^d} f(u) \exp\left(-2\pi^2 |u|_{\mathbb{R}^d}^2 - 2\pi\sqrt{-1}(u, \xi)_{\mathbb{R}^d}\right) du \quad (\xi \in \mathbb{R}^d).$$

It is easy to see

$$\mathfrak{F}(1)(\xi) = (2\pi)^{-d/2} \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) \quad (\xi \in \mathbb{R}^d), \quad (6.10)$$

and

$$u^\alpha \exp\left(-2\pi\sqrt{-1}(u, \xi)_{\mathbb{R}^d}\right) = \frac{1}{(-2\pi\sqrt{-1})^{|\alpha|}} \partial_\xi^\alpha \exp\left(-2\pi\sqrt{-1}(u, \xi)_{\mathbb{R}^d}\right),$$

where  $\partial_\xi^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} = (\frac{\partial}{\partial \xi_1})^{\alpha_1} \cdots (\frac{\partial}{\partial \xi_d})^{\alpha_d}$  for  $\xi = (\xi_1, \dots, \xi_d)$ ,  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ . Differentiating of the left-hand side of (6.10) with respect to  $\xi$  gives

$$\mathfrak{F}(u^\alpha)(\xi) = \frac{1}{(2\pi)^{d/2} (-2\pi\sqrt{-1})^{|\alpha|}} \partial_\xi^\alpha \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) \quad (\xi \in \mathbb{R}^d). \quad (6.11)$$

On the other hand, we also have

$$\begin{aligned} \partial_i \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) &= -\xi_i \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\ \partial_i \partial_j \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) &= (\xi_i \xi_j - \delta_{ij}) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\ \partial_i \partial_j \partial_k \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) &= (-\xi_i \xi_j \xi_k + \delta_{ij} \xi_k + \delta_{jk} \xi_i + \delta_{ki} \xi_j) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\ \partial_i \partial_j \partial_k \partial_l \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right) &= (\xi_i \xi_j \xi_k \xi_l - \delta_{ij} \xi_k \xi_l - \delta_{jk} \xi_l \xi_i - \delta_{kl} \xi_i \xi_j - \delta_{li} \xi_j \xi_k \\ &\quad - \delta_{ik} \xi_j \xi_l - \delta_{jl} \xi_i \xi_k + \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{li} \delta_{jk}) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right). \end{aligned}$$

Combining these identities with (6.11), we obtain

$$\begin{aligned}
\mathfrak{F}(u_i)(\xi) &= -(2\pi)^{-d/2} \frac{\sqrt{-1}\xi_i}{2\pi} \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\
\mathfrak{F}(u_i u_j)(\xi) &= -(2\pi)^{-d/2} \frac{\xi_i \xi_j - \delta_{ij}}{4\pi^2} \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\
\mathfrak{F}(u_i u_j u_k)(\xi) &= -(2\pi)^{-d/2} \frac{\sqrt{-1}}{8\pi^3} \left(-\xi_i \xi_j \xi_k + \delta_{ij} \xi_k + \delta_{jk} \xi_i + \delta_{ki} \xi_j\right) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right), \\
\mathfrak{F}(u_i u_j u_k u_l)(\xi) &= (2\pi)^{-d/2} \frac{1}{16\pi^4} \left(\xi_i \xi_j \xi_k \xi_l - \delta_{ij} \xi_k \xi_l - \delta_{jk} \xi_l \xi_i - \delta_{kl} \xi_i \xi_j - \delta_{li} \xi_j \xi_k \right. \\
&\quad \left. - \delta_{ik} \xi_j \xi_l - \delta_{jl} \xi_i \xi_k + \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{li} \delta_{jk}\right) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right).
\end{aligned}$$

Repeating this argument twice, we further obtain

$$\begin{aligned}
\mathfrak{F}(u_i u_j u_k u_l u_m u_n)(\xi) &= -(2\pi)^{-d/2} \frac{1}{64\pi^6} \left(\xi_i \xi_j \xi_k \xi_l \xi_m \xi_n - \mathfrak{f}_4(i, j, k, l, m, n)(\xi) \right. \\
&\quad \left. + \mathfrak{f}_2(i, j, k, l, m, n)(\xi) - \mathfrak{g}(i, j, k, l, m, n)\right) \exp\left(-\frac{|\xi|_{\mathbb{R}^d}^2}{2}\right),
\end{aligned}$$

where  $\mathfrak{f}_r(i, j, k, l, m, n)(\xi)$  ( $r = 2, 4$ ) is a homogeneous polynomial of degree  $r$  in the variables  $\xi_i, \xi_j, \dots, \xi_n$  and the constant  $\mathfrak{g}(i, j, k, l, m, n)$  is given by

$$\begin{aligned}
\mathfrak{g}(i, j, k, l, m, n) &= \delta_{ij}(\delta_{kl} \delta_{mn} + \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}) + \delta_{ik}(\delta_{jl} \delta_{mn} + \delta_{jm} \delta_{ln} + \delta_{jn} \delta_{lm}) \\
&\quad + \delta_{il}(\delta_{jk} \delta_{mn} + \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km}) + \delta_{im}(\delta_{jk} \delta_{ln} + \delta_{jl} \delta_{kn} + \delta_{jn} \delta_{kl}) \\
&\quad + \delta_{in}(\delta_{jk} \delta_{lm} + \delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl}).
\end{aligned}$$

### 6.3 Proof of Theorem 2.5

Since  $p(n, x, y) = 0$  for  $x \in A_i, y \in A_j$  and  $n \neq Kl + j - i$  is obvious, we only consider the case  $x \in A_i, y \in A_j$  and  $n = Kl + j - i$ . For later use, we take an extension of  $\lambda_\omega(1) = -\log \mu_0(\chi_\omega)$  to the whole space  $\text{Hom}(\Gamma, \mathbb{R})$  such that  $\text{Re} \lambda_\omega(1) \geq b \|\omega\|^2$  for some constant  $b > 0$ . We also extend  $\phi_\omega = \phi_{0, \chi_\omega}$  and  $\psi_\omega = \psi_{0, \chi_\omega}$  to smooth compact supported functions on  $\text{Hom}(\Gamma, \mathbb{R})$ , and set  $\mathfrak{p}(\omega; x, y) = \phi_\omega(\pi(x)) \overline{\psi_\omega(\pi(y))}$ . Since  $\mathfrak{p}$  has compact support in the variable  $\omega$ , it holds

$$\|\mathfrak{p}\|_\infty := \sup \{ |\mathfrak{p}(\omega; x, y)| : \omega \in \text{Hom}(\Gamma, \mathbb{R}), x, y \in V \} < \infty.$$

We take an orthonormal basis  $\{\omega_1, \dots, \omega_d\}$  of  $\text{Hom}(\Gamma, \mathbb{R}) (\subset H^1(X_0, \mathbb{R}))$  and write  $\omega = u_1 \omega_1 + \dots + u_d \omega_d$ . Then the normalized Haar measure  $d\chi$  on  $\widehat{\Gamma}$  is written as

$$d\chi = d\chi_\omega = \text{vol}(\widehat{\Gamma})^{-1} d\omega = \text{vol}(\text{Alb}^\Gamma) d\omega,$$

where  $d\omega = du = du_1 \cdots du_d$  is the Lebesgue measure on  $\text{Hom}(\Gamma, \mathbb{R})$ .



We divide the proof of Theorem 2.5 into several steps.

**Step 1.** Recall the integral representation (6.3). As mentioned in Subsection 6.1, we may choose a constant  $0 < \eta < 1$  and a sufficiently small neighborhood  $U(\mathbf{1})$  of the trivial character  $\chi = \mathbf{1}$  such that  $|\mu_0(\chi)| < \eta$  for all  $\chi \in \widehat{\Gamma} \setminus U(\mathbf{1})$ . By (6.3), (6.4) and (6.5), we have

$$\begin{aligned}
p(n, x, y) &= K \text{vol}(\text{Alb}^\Gamma) \int_{\text{Hom}(\Gamma, \mathbb{R})} \exp(-n\lambda_\omega(1)) \phi_\omega(\pi(x)) \overline{\psi_\omega(\pi(y))} \\
&\quad \times \exp\left(-2\pi\sqrt{-1}\langle\omega, \Phi_0(y) - \Phi_0(x)\rangle\right) d\omega + C\eta^n \\
&= K \text{vol}(\text{Alb}^\Gamma) n^{-d/2} \int_{\text{Hom}(\Gamma, \mathbb{R})} \exp\left(-n\lambda_{\frac{\omega}{\sqrt{n}}}(1)\right) \phi_{\omega/\sqrt{n}}(\pi(x)) \overline{\psi_{\omega/\sqrt{n}}(\pi(y))} \\
&\quad \times \exp\left(-2\pi\sqrt{-1}\langle\frac{\omega}{\sqrt{n}}, \Phi_0(y) - \Phi_0(x)\rangle\right) d\omega + C\eta^n \\
&=: K \text{vol}(\text{Alb}^\Gamma) n^{-d/2} I(n) + C\eta^n
\end{aligned}$$

for some positive constant  $C$  independent of  $x$  and  $y$ . For simplicity, we now set

$$\mathfrak{d}_n(x, y; \rho_\mathbb{R}(\gamma_p)) := n^{-1/2}(\Phi_0(y) - \Phi_0(x) - n\rho_\mathbb{R}(\gamma_p)).$$

By expanding  $n\lambda_{\omega/\sqrt{n}}(1)$  as

$$\begin{aligned}
n\lambda_{\omega/\sqrt{n}}(1) &= n \sum_{k=0}^4 \frac{1}{k!} \lambda_{\omega/\sqrt{n}}^{(k)}(0) + nO\left(\frac{\|\omega\|^5}{n^{5/2}}\right) \\
&= n\left(-2\pi\sqrt{-1}\langle\gamma_p, \frac{\omega}{\sqrt{n}}\rangle + 2\pi^2\left\|\frac{\omega}{\sqrt{n}}\right\|^2\right) + n\left(\frac{1}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) + \frac{1}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \\
&\quad + nO\left(\frac{\|\omega\|^5}{n^{5/2}}\right),
\end{aligned}$$

we have

$$\begin{aligned}
I(n) &= \int_{\text{Hom}(\Gamma, \mathbb{R})} e^{-2\pi^2\|\omega\|^2} \exp\left(-\frac{n}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) - \frac{n}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \\
&\quad \times \exp\left\{O\left(\frac{\|\omega\|^5}{n^{3/2}}\right)\right\} \mathfrak{p}(\omega/\sqrt{n}; x, y) \exp\left(-2\pi\sqrt{-1}\langle\omega, \mathfrak{d}_n(\gamma_p; x, y)\rangle\right) d\omega \\
&= \int_{\|\omega\| \leq n^{1/6}} e^{-2\pi^2\|\omega\|^2} \exp\left(-\frac{n}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) - \frac{n}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \\
&\quad \times \exp\left\{O\left(\frac{\|\omega\|^5}{n^{3/2}}\right)\right\} \mathfrak{p}(\omega/\sqrt{n}; x, y) \exp\left(-2\pi\sqrt{-1}\langle\omega, \mathfrak{d}_n(\gamma_p; x, y)\rangle\right) d\omega \\
&\quad + \int_{\|\omega\| > n^{1/6}} \exp\left(-n\lambda_{\frac{\omega}{\sqrt{n}}}(1)\right) \mathfrak{p}(\omega/\sqrt{n}; x, y) \\
&\quad \times \exp\left(-2\pi\sqrt{-1}\langle\frac{\omega}{\sqrt{n}}, \Phi_0(y) - \Phi_0(x)\rangle\right) d\omega \\
&=: I_1(n) + I_2(n).
\end{aligned}$$

Recalling  $n\operatorname{Re}\lambda_{\omega/\sqrt{n}}(1) \geq b\|\omega\|^2$  ( $\omega \in \operatorname{Hom}(\Gamma, \mathbb{R})$ ), we obtain

$$|I_2(n)| \leq \|\mathbf{p}\|_\infty \int_{\|\omega\| > n^{1/6}} e^{-b\|\omega\|^2} d\omega$$

Thus  $I_2(n)$  converges to 0 as  $n \rightarrow \infty$  exponentially fast uniformly for  $x, y \in V$ .

**Step 2.** We divide the integral  $I_1(n)$  into

$$\begin{aligned} I_1(n) &= \int_{\|\omega\| \leq n^{1/6}} e^{-2\pi^2\|\omega\|^2} \exp\left(-\frac{n}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) - \frac{n}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \\ &\quad \times \mathbf{p}(\omega/\sqrt{n}; x, y) \exp\left(-2\pi\sqrt{-1}\langle\omega, \mathfrak{d}_n(\gamma_p; x, y)\rangle\right) d\omega \\ &\quad + \int_{\|\omega\| \leq n^{1/6}} e^{-2\pi^2\|\omega\|^2} \exp\left(-\frac{n}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) - \frac{n}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \mathbf{p}(\omega/\sqrt{n}; x, y) \\ &\quad \times \left(\exp\left\{O\left(\frac{\|\omega\|^5}{n^{3/2}}\right)\right\} - 1\right) \exp\left(-2\pi\sqrt{-1}\langle\omega, \mathfrak{d}_n(\gamma_p; x, y)\rangle\right) d\omega \\ &=: I_3(n) + I_4(n). \end{aligned}$$

Thanks to Lemma 6.6, we find

$$n\lambda_{\omega/\sqrt{n}}^{(3)}(0) = nO\left(\frac{\|\omega\|}{n^{1/2}}\right)^3 = O(1), \quad n\lambda_{\omega/\sqrt{n}}^{(4)}(0) = nO\left(\frac{\|\omega\|}{n^{1/2}}\right)^4 = O(n^{-1/2})$$

and

$$\begin{aligned} \left|\exp\left\{O\left(\frac{\|\omega\|^5}{n^{3/2}}\right)\right\} - 1\right| &= \exp\left\{O\left(\frac{\|\omega\|^5}{n^{3/2}}\right)\right\} O\left(\frac{\|\omega\|^5}{n^{3/2}}\right) \\ &= \exp\left(O(n^{-2/3})\right) O\left(\frac{\|\omega\|^5}{n^{3/2}}\right). \end{aligned}$$

provided  $\|\omega\| \leq n^{1/6}$ . Thus we can estimate  $I_4(n)$  as

$$|I_4(n)| \leq C\|\mathbf{p}\|_\infty \int_{\operatorname{Hom}(\Gamma, \mathbb{R})} O\left(\frac{\|\omega\|^5}{n^{3/2}}\right) e^{-2\pi^2\|\omega\|^2} d\omega = O(n^{-3/2}).$$

**Step 3.** As a direct consequence of Lemma 6.6, we easily obtain

**Lemma 6.7** *There exist real constants  $\mathbf{q}_\alpha = \mathbf{q}_\alpha(\pi(x), \pi(y); \gamma_p)$  ( $\alpha = (\alpha_1, \dots, \alpha_r) \in \{1, \dots, d\}^r$ ,  $|\alpha| = r = 1, \dots, 4$ ) such that  $\mathbf{q}_{\sigma(\alpha)} = \mathbf{q}_\alpha$  ( $\sigma \in \mathcal{S}_r$ ) and*

$$\begin{aligned} |V_0|^{-1/2} \overline{\psi'_{\omega(u)/\sqrt{n}}(\pi(y))} &= \frac{\sqrt{-1}}{n^{1/2}} \sum_{i=1}^d \mathbf{q}_i u_i =: \frac{\sqrt{-1}}{n^{1/2}} \mathcal{Q}_1(u), \\ |V_0|^{1/2} \phi''_{\omega(u)/\sqrt{n}}(\pi(x)) + |V_0|^{-1/2} \overline{\psi''_{\omega(u)/\sqrt{n}}(\pi(y))} \\ &= \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2 \sum_{i,j=1}^d \mathbf{q}_{ij} u_i u_j =: \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2 \mathcal{Q}_2(u), \end{aligned}$$

$$\begin{aligned}\lambda_{\omega(u)/n^{1/2}}^{(3)}(0) &= \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^3 \sum_{i,j,k=1}^d \mathfrak{q}_{ijk} u_i u_j u_k =: \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^3 \mathcal{Q}_3(u), \\ \lambda_{\omega(u)/n^{1/2}}^{(4)}(0) &= \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^4 \sum_{i,j,k,l=1}^d \mathfrak{q}_{ijkl} u_i u_j u_k u_l =: \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^4 \mathcal{Q}_4(u)\end{aligned}$$

hold for all  $\omega = \omega(u) = u_1 \omega_1 + \cdots + u_d \omega_d \in \text{Hom}(\Gamma, \mathbb{R})$ .

Then, under the condition  $\|\omega\| \leq n^{1/6}$ , we can expand as

$$\begin{aligned}& \exp\left(-\frac{n}{6}\lambda_{\omega/\sqrt{n}}^{(3)}(0) - \frac{n}{24}\lambda_{\omega/\sqrt{n}}^{(4)}(0)\right) \\ &= 1 - \left(\frac{\lambda_{\omega/\sqrt{n}}^{(3)}(0)}{6} + \frac{\lambda_{\omega/\sqrt{n}}^{(4)}(0)}{24}\right)n + \frac{1}{2}\left(\frac{\lambda_{\omega/\sqrt{n}}^{(3)}(0)}{6} + \frac{\lambda_{\omega/\sqrt{n}}^{(4)}(0)}{24}\right)^2 n^2 \\ &\quad + \frac{1}{6}\exp\left(O(1) + O(n^{-2/3})\right)\left(\frac{\lambda_{\omega/\sqrt{n}}^{(3)}(0)}{6} + \frac{\lambda_{\omega/\sqrt{n}}^{(4)}(0)}{24}\right)^3 n^3 \\ &= 1 + \left(\frac{\sqrt{-1}}{n^{1/2}}\right)\frac{\mathcal{Q}_3(u)}{6} + \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2\left(\frac{\mathcal{Q}_4(u)}{24} + \frac{\mathcal{Q}_3(u)^2}{72}\right) + O\left(\frac{|u|_{\mathbb{R}^d}^9}{n^{3/2}}\right),\end{aligned}\tag{6.12}$$

and

$$\begin{aligned}\mathfrak{p}(\omega/\sqrt{n}; x, y) &= (\phi_0(\pi(x)) + \phi'_0(\pi(x)) + \cdots)(\overline{\psi_0(\pi(y))} + \overline{\psi'_0(\pi(y))} + \cdots) \\ &= m(\pi(y)) + |V_0|^{-1/2}\overline{\psi'_{\omega/\sqrt{n}}(\pi(y))} \\ &\quad + \frac{1}{2}\left(|V_0|^{1/2}\phi_{\omega/\sqrt{n}}^{(2)}(\pi(x))m(\pi(y)) + |V_0|^{-1/2}\overline{\psi_{\omega/\sqrt{n}}^{(2)}(\pi(y))}\right) \\ &\quad + O\left(\frac{\|\omega\|^3}{n^{3/2}}\right) \\ &= m(\pi(y)) + \left(\frac{\sqrt{-1}}{n^{1/2}}\right)\mathcal{Q}_1(u) + \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2\frac{\mathcal{Q}_2(u)}{2} + O\left(\frac{|u|_{\mathbb{R}^d}^3}{n^{3/2}}\right).\end{aligned}\tag{6.13}$$

Multiplying (6.12) and (6.13) together, we obtain

$$\begin{aligned}I_3(n) &= \int_{|u|_{\mathbb{R}^d} \leq n^{1/6}} \sum_{i=0}^2 \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^i \mathcal{A}_i(u) \\ &\quad \times \exp\left(-2\pi^2|u|_{\mathbb{R}^d}^2 - 2\pi\sqrt{-1}(u, \mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p)))_{\mathbb{R}^d}\right) du + O(n^{-3/2}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_0(u) &= m(\pi(y)), \\ \mathcal{A}_1(u) &= \mathcal{Q}_1(u) + \frac{m(\pi(y))}{6}\mathcal{Q}_3(u), \\ \mathcal{A}_2(u) &= \frac{1}{2}\mathcal{Q}_2(u) + \frac{1}{6}\mathcal{Q}_1(u)\mathcal{Q}_3(u) + m(\pi(y))\left(\frac{\mathcal{Q}_4(u)}{24} + \frac{\mathcal{Q}_3(u)^2}{72}\right).\end{aligned}$$

We also find that the integral

$$\int_{|u|_{\mathbb{R}^d} > n^{1/6}} \sum_{i=0}^2 \left( \frac{\sqrt{-1}}{n^{1/2}} \right)^i \mathcal{A}_i(u) \exp \left( -2\pi^2 |u|_{\mathbb{R}^d}^2 - 2\pi \sqrt{-1} (u, \mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p)))_{\mathbb{R}^d} \right) du$$

converges to 0 as  $n \rightarrow \infty$  exponentially fast in the same way as Step 1.

Putting it all together, we now obtain

$$\begin{aligned} I(n) &= \int_{\mathbb{R}^d} \sum_{i=0}^2 \left( \frac{\sqrt{-1}}{n^{1/2}} \right)^i \mathcal{A}_i(u) \\ &\quad \times \exp \left( -2\pi^2 |u|_{\mathbb{R}^d}^2 - 2\pi \sqrt{-1} (u, \mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p)))_{\mathbb{R}^d} \right) du + O(n^{-3/2}) \\ &= \sum_{i=0}^2 \left( \frac{\sqrt{-1}}{n^{1/2}} \right)^i \mathfrak{F}(\mathcal{A}_i)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) + O(n^{-3/2}). \end{aligned} \quad (6.14)$$

**Step 4.** We are now in a position to calculate  $\left( \frac{\sqrt{-1}}{n^{1/2}} \right)^i \mathfrak{F}(\mathcal{A}_i)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p)))$  ( $n = 0, 1, 2$ ). It follows directly from (6.10) that

$$\mathfrak{F}(\mathcal{A}_0)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) = (2\pi)^{-d/2} m(\pi(y)) \exp \left( -\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right).$$

Using (6.11) and noting the condition  $|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0} \leq Cn^{1/6}$ , we have

$$\begin{aligned} &\left( \frac{\sqrt{-1}}{n^{1/2}} \right) \mathfrak{F}(\mathcal{A}_1)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) \\ &= (2\pi)^{-d/2} m(\pi(y)) \exp \left( -\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right) \\ &\quad \times \left\{ \left( \frac{1}{2\pi} m(\pi(y))^{-1} \sum_{i=1}^d \mathfrak{q}_i (\Phi_0(y) - \Phi_0(x) - \rho_{\mathbb{R}}(\gamma_p))_i \right. \right. \\ &\quad \left. \left. + \frac{1}{16\pi^3} \sum_{i,j=1}^d \mathfrak{q}_{ij} (\Phi_0(y) - \Phi_0(x) - \rho_{\mathbb{R}}(\gamma_p))_j \right) n^{-1} + O(n^{-3/2}) \right\}. \end{aligned}$$

In the same way, we also obtain

$$\begin{aligned} &\left( \frac{\sqrt{-1}}{n^{1/2}} \right)^2 \frac{1}{2} \mathfrak{F}(\mathcal{Q}_2)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) \\ &= (2\pi)^{-d/2} m(\pi(y)) \exp \left( -\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n} \right) \\ &\quad \times \left\{ \left( -\frac{m(\pi(y))^{-1}}{8\pi^2} \sum_{i=1}^d \mathfrak{q}_{ii} \right) n^{-1} + O(n^{-5/3}) \right\}, \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2 \frac{1}{6} \mathfrak{F}(\mathcal{Q}_1 \mathcal{Q}_3)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) \\
&= (2\pi)^{-d/2} m(\pi(y)) \exp\left(-\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n}\right) \\
&\quad \times \left\{ \left(-\frac{m(\pi(y))^{-1}}{32\pi^4} \sum_{i,j=1}^d \mathfrak{q}_i \mathfrak{q}_{ijj}\right) n^{-1} + O(n^{-5/3}) \right\}, \\
& \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2 \frac{m(\pi(y))}{24} \mathfrak{F}(\mathcal{Q}_4)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) \\
&= (2\pi)^{-d/2} m(\pi(y)) \exp\left(-\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n}\right) \\
&\quad \times \left\{ \left(-\frac{1}{128\pi^4} \sum_{i,j=1}^d \mathfrak{q}_{ijj}\right) n^{-1} + O(n^{-5/3}) \right\}, \\
& \left(\frac{\sqrt{-1}}{n^{1/2}}\right)^2 \frac{m(\pi(y))}{72} \mathfrak{F}(\mathcal{Q}_3^2)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) \\
&= (2\pi)^{-d/2} m(\pi(y)) \exp\left(-\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n}\right) \\
&\quad \times \left\{ \left(-\frac{5}{1536\pi^6} \sum_{i,j,k=1}^d \mathfrak{q}_{ijj} \mathfrak{q}_{jkk}\right) n^{-1} + O(n^{-5/3}) \right\}.
\end{aligned}$$

Recalling  $m(\pi(y)) = m(y)$  and summarizing all above arguments, we complete the proof of Theorem 2.5. Moreover, we also obtain the explicit expression of the coefficient  $a_1 = a_1(\pi(x), \pi(y), \gamma_p; \Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p))$  in (2.8) as follows:

**Theorem 6.8**

$$\begin{aligned}
& a_1(\pi(x), \pi(y), \gamma_p; \Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)) \\
&= \frac{m(\pi(y))^{-1}}{2\pi} \sum_{i=1}^d \mathfrak{q}_i (\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p))_i \\
&\quad + \frac{1}{16\pi^3} \sum_{i,j=1}^d \mathfrak{q}_{ij} (\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p))_j - \frac{m(\pi(y))^{-1}}{8\pi^2} \sum_{i=1}^d \mathfrak{q}_{ii} \\
&\quad - \frac{m(\pi(y))^{-1}}{32\pi^4} \sum_{i,j=1}^d \mathfrak{q}_i \mathfrak{q}_{ijj} - \frac{1}{128\pi^4} \sum_{i,j=1}^d \mathfrak{q}_{ijj} - \frac{5}{1536\pi^6} \sum_{i,j,k=1}^d \mathfrak{q}_{ijj} \mathfrak{q}_{jkk},
\end{aligned}$$

where the coefficients  $\mathfrak{q}_\alpha = \mathfrak{q}_\alpha(\pi(x), \pi(y); \gamma_p)$  ( $\alpha = (\alpha_1, \dots, \alpha_r) \in \{1, \dots, d\}^r, r = 1, \dots, 4$ ) are given in Lemma 6.7.

**Remark 6.9** As we observed in Subsection 6.1, the coefficients  $\mathfrak{q}_\alpha$  ( $|\alpha| = 1, 3$ ) are equal

to 0 provided the random walk is  $m$ -symmetric (i.e.,  $\gamma_p = 0$ ). In that case, we have

$$\begin{aligned} & a_1(\pi(x), \pi(y); \Phi_0(y) - \Phi_0(x)) \\ &= \frac{1}{16\pi^3} \sum_{i,j=1}^d \mathfrak{q}_{ij} (\Phi_0(y) - \Phi_0(x))_j - \frac{m(\pi(y))^{-1}}{8\pi^2} \sum_{i=1}^d \mathfrak{q}_{ii} - \frac{1}{128\pi^4} \sum_{i,j=1}^d \mathfrak{q}_{iijj}. \end{aligned}$$

Before closing this subsection, we should mention that

$$I(n) = \mathfrak{F}(\mathcal{A}_0)(\mathfrak{d}_n(x, y; \rho_{\mathbb{R}}(\gamma_p))) + O(n^{-1/2}) \quad (6.15)$$

also holds uniformly for  $x, y \in V$  in Step 3 of the above proof of Theorem 2.5. Then by replacing (6.14) by (6.15), we easily obtain the following LCLT:

**Corollary 6.10 (Sunada [22])** *Suppose that the random walk  $\{w_n\}_{n=0}^\infty$  on  $X$  is irreducible with period  $K$ . Let  $V = \coprod_{k=0}^{K-1} A_k$  be the corresponding  $K$ -partition of  $V$ . Then for any  $x \in A_i$  and  $y \in A_j$  ( $0 \leq i, j \leq K-1$ ), we have*

$$p(n, x, y) = 0 \quad (n \neq Kl + j - i),$$

and

$$(2\pi n)^{d/2} p(n, x, y) m(y)^{-1} - K \text{vol}(\text{Alb}^\Gamma) \exp\left(-\frac{|\Phi_0(y) - \Phi_0(x) - n\rho_{\mathbb{R}}(\gamma_p)|_{g_0}^2}{2n}\right) \rightarrow 0$$

as  $n = Kl + j - i \rightarrow \infty$  uniformly for  $x$  and  $y$ .

## 6.4 Application: Another approach to the CLT of the first kind

In this subsection, applying the LCLT (Corollary 6.10), we give another proof of (2.4) in Theorem 2.1 under the irreducibility condition of the random walk on  $X$ . For simplicity of the argument, we only consider the case  $K = 1$ . (In other cases, the proof goes through in a very similar way with a slight modification.)

First of all, we take a sequence  $\{(x_n, \mathbf{z}_n)\}_{n=1}^\infty$  in  $V \times H_1(X_0, \mathbb{R})$  satisfying

$$\lim_{n \rightarrow \infty} n^{-1/2} (\Phi_0(x_n) - \rho_{\mathbb{R}}(\mathbf{z}_n)) = \mathbf{x} \in \Gamma \otimes \mathbb{R}.$$

By (4.5), it is enough to show that for any  $f \in C_0^\infty(\Gamma \otimes \mathbb{R})$ ,

$$\lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, \mathbf{z}_n) = e^{-t\Delta/2} f(\mathbf{x}),$$

that is,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{y \in V} p([nt], x_n, y) f(n^{-1/2} (\Phi_0(y) - \rho_{\mathbb{R}}(\mathbf{z}_n) - [nt]\rho_{\mathbb{R}}(\gamma_p))) \\ &= \int_{\Gamma \otimes \mathbb{R}} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{2t}\right) f(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

We set

$$\mathbf{x}_n := n^{-1/2}(\Phi_0(x_n) - \rho_{\mathbb{R}}(\mathbf{z}_n)), \quad \mathbf{y}_n(y) := n^{-1/2}(\Phi_0(y) - \rho_{\mathbb{R}}(\mathbf{z}_n) - [nt]\rho_{\mathbb{R}}(\gamma_p)) \quad (y \in V).$$

By the LCLT (2.7), for an arbitrary small  $\varepsilon > 0$ , there is a positive integer  $n_0$  such that for any  $n \geq n_0$ ,

$$\left| (2\pi[nt])^{d/2} p([nt], x_n, y) m(y)^{-1} - \text{vol}(\text{Alb}^\Gamma) \exp\left(-\frac{n|\mathbf{x}_n - \mathbf{y}_n(y)|_{g_0}^2}{2[nt]}\right) \right| < \varepsilon$$

uniformly for  $x_n, y \in V$ . Dividing both sides by  $(2\pi[nt])^{d/2} m(y)^{-1}$ , for any  $\varepsilon' > 0$ , there exists  $n_1 \in \mathbb{N}$  such that for any  $n \geq n_1$ ,

$$\left| p([nt], x_n, y) - \frac{\text{vol}(\text{Alb}^\Gamma) m(y)}{(2\pi[nt])^{d/2}} \exp\left(-\frac{n|\mathbf{x}_n - \mathbf{y}_n(y)|_{g_0}^2}{2[nt]}\right) \right| < \varepsilon' n^{-d/2}.$$

Then we obtain

$$\begin{aligned} & \left| \sum_{y \in V} \left\{ p([nt], x_n, y) f(\mathbf{y}_n(y)) - \frac{\text{vol}(\text{Alb}^\Gamma) m(y)}{(2\pi[nt])^{d/2}} \exp\left(-\frac{n|\mathbf{x}_n - \mathbf{y}_n(y)|_{g_0}^2}{2[nt]}\right) f(\mathbf{y}_n(y)) \right\} \right| \\ & \leq \sum_{y \in V} \left| p([nt], x_n, y) - \frac{\text{vol}(\text{Alb}^\Gamma) m(y)}{(2\pi[nt])^{d/2}} \exp\left(-\frac{n|\mathbf{x}_n - \mathbf{y}_n(y)|_{g_0}^2}{2[nt]}\right) \right| |f(\mathbf{y}_n(y))| \\ & \leq \varepsilon' n^{-d/2} \sum_{y \in V} |f(\mathbf{y}_n(y))|. \end{aligned}$$

Since the support of  $f$  is compact,

$$\begin{aligned} n^{-d/2} \sum_{y \in V} |f(\mathbf{y}_n(y))| &= n^{-d/2} \sum_{y_0 \in \mathcal{F}} \sum_{\sigma \in \Gamma} \left| f\left(\mathbf{y}_n(y_0) + \frac{\sigma}{\sqrt{n}}\right) \right| \\ &\rightarrow \frac{|V_0|}{\text{vol}(\text{Alb}^\Gamma)} \int_{\Gamma \otimes \mathbb{R}} |f(\mathbf{y})| d\mathbf{y} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Then we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, \mathbf{z}_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2\pi[nt])^{d/2}} \sum_{y \in V} m(y) \text{vol}(\text{Alb}^\Gamma) \exp\left(-\frac{n}{2[nt]} |\mathbf{y}_n(y) - \mathbf{x}_n|_{g_0}^2\right) f(\mathbf{y}_n(y)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2\pi nt)^{d/2}} \sum_{y \in V} m(y) \text{vol}(\text{Alb}^\Gamma) \exp\left(-\frac{1}{2t} |\mathbf{y}_n(y) - \mathbf{x}_n|_{g_0}^2\right) f(\mathbf{y}_n(y)). \end{aligned}$$

By [11, pp. 655], for any  $\varepsilon > 0$ , there exists  $n_2 \in \mathbb{N}$  such that for any  $n \geq n_2$ ,

$$\left| \exp\left(-\frac{1}{2t} |\mathbf{y}_n(y) - \mathbf{x}_n|_{g_0}^2\right) - \exp\left(-\frac{1}{2t} |\mathbf{y}_n(y) - \mathbf{x}|_{g_0}^2\right) \right| < \varepsilon$$

uniformly for  $y \in V$ .

Thus we conclude

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \mathcal{L}_{\gamma_p}^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, \mathbf{z}_n) \\
&= \lim_{n \rightarrow \infty} \frac{1}{(2\pi nt)^{d/2}} \sum_{y \in V} m(y) \text{vol}(\text{Alb}^\Gamma) \exp\left(-\frac{1}{2t} |\mathbf{y}_n(y) - \mathbf{x}|_{g_0}^2\right) f(\mathbf{y}_n(y)) \\
&= \frac{1}{(2\pi t)^{d/2}} \lim_{n \rightarrow \infty} \sum_{y_0 \in \mathcal{F}} m(y_0) \sum_{\sigma \in \Gamma} \frac{\text{vol}(\text{Alb}^\Gamma)}{n^{d/2}} \\
&\quad \times f\left(\mathbf{y}_n(y_0) + \frac{\sigma}{\sqrt{n}}\right) \exp\left(-\frac{|\mathbf{x} - (\mathbf{y}_n(y_0) + \frac{\sigma}{\sqrt{n}})|_{g_0}^2}{2n}\right) \\
&= \frac{1}{(2\pi t)^{d/2}} \int_{\Gamma \otimes \mathbb{R}} f(\mathbf{y}) \exp\left(-\frac{|\mathbf{x} - \mathbf{y}|_{g_0}^2}{2t}\right) d\mathbf{y},
\end{aligned}$$

whence (2.4) follows.

## 7 Examples of the modified standard realization

In this final section, we give several examples of the modified standard realization of crystal lattices associated with non-symmetric random walks. See [15, 11, 12] for the symmetric case. Here we write  $\hat{\alpha} := \alpha + \alpha'$  and  $\check{\alpha} := \alpha - \alpha'$  for two numbers  $\alpha, \alpha'$ , and denote  $\sigma \otimes 1 \in \Gamma \otimes \mathbb{R}$  by the same symbol  $\sigma$  for simplicity of notation.

### 7.1 The 2-dimensional square lattice

Let  $X = (V, E)$  be the 2-dimensional square lattice graph. Namely,  $V = \mathbb{Z}^2$  and

$$E = \{(\mathbf{x}, \mathbf{y}) \in V \times V \mid \mathbf{y} - \mathbf{x} \in \{\pm(1, 0), \pm(0, 1)\}\}.$$

We consider a random walk on  $X$  whose transition probability is given by

$$\begin{aligned}
p((\mathbf{x}, \mathbf{x} + (1, 0))) &= \alpha, & p((\mathbf{x}, \mathbf{x} - (1, 0))) &= \alpha', \\
p((\mathbf{x}, \mathbf{x} + (0, 1))) &= \beta, & p((\mathbf{x}, \mathbf{x} - (0, 1))) &= \beta'
\end{aligned}$$

for every  $\mathbf{x}$ . Here we assume

$$\alpha, \alpha', \beta, \beta' \geq 0 \quad \text{and} \quad \alpha + \alpha' + \beta + \beta' = 1.$$

It is easy to see that  $X$  is invariant under the  $\mathbb{Z}^2$ -action generated by  $\sigma_1(\mathbf{x}) = \mathbf{x} + (1, 0)$  and  $\sigma_2(\mathbf{x}) = \mathbf{x} + (0, 1)$ . Its quotient  $X_0$  is a 2-bouquet graph  $X_0 = (V_0, E_0) = (\{x_0\}, \{e_1, e_2\})$  (see Figure 4).



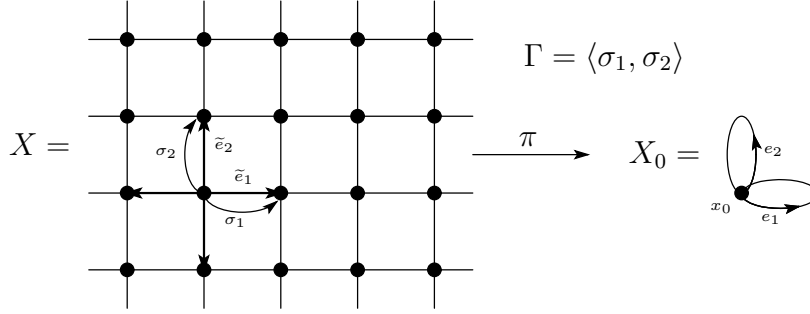


Figure 4: Square lattice graph and its quotient

By definition, we have  $\gamma_p = \check{\alpha}[e_1] + \check{\beta}[e_2] \in H^1(X_0, \mathbb{R})$  and  $m(x_0) = 1$ . Since  $X_0$  is a bouquet graph, the exterior derivative  $d : C^0(X_0, \mathbb{R}) \rightarrow C^1(X_0, \mathbb{R})$  is the 0-map. Then

$$H^1(X_0, \mathbb{R}) \cong \mathcal{H}^1(X_0) = C^1(X_0, \mathbb{R}).$$

We define the canonical surjective linear map  $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R}$  by

$$\rho_{\mathbb{R}}([e_i]) := \sigma_i \quad (i = 1, 2).$$

We are going to determine the modified standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ . Let  $\tilde{e}_i$  ( $i = 1, 2$ ) be a lift of  $e_i$  to  $X$ , and we set  $\Phi_0(o(\tilde{e}_1)) = \Phi_0(o(\tilde{e}_2)) = \mathbf{0}$ . Noting that the asymptotic direction is given by  $\rho_{\mathbb{R}}(\gamma_p) = \check{\alpha}\sigma_1 + \check{\beta}\sigma_2$ , we easily see that

$$\Phi_0(t(\tilde{e}_1)) = \sigma_1, \quad \Phi_0(t(\tilde{e}_2)) = \sigma_2$$

is the modified harmonic realization. We next take the dual basis  $\{\omega_1, \omega_2\} \subset \text{Hom}(\Gamma, \mathbb{R})$  of  $\{\sigma_1, \sigma_2\}$ . Namely,  $\omega_i[\sigma_j]_{\Gamma \otimes \mathbb{R}} = \delta_{ij}$  ( $i, j = 1, 2$ ). Because  $\omega \in \text{Hom}(\Gamma, \mathbb{R})$  is identified with  $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in H^1(X_0, \mathbb{R})$ , we have

$$\omega_i(e_j) = {}_{H^1(X_0, \mathbb{R})} \langle {}^t\rho_{\mathbb{R}}(\omega_i), [e_j] \rangle_{H_1(X_0, \mathbb{R})} = {}_{\text{Hom}(\Gamma, \mathbb{R})} \langle \omega_i, \rho_{\mathbb{R}}([e_j]) \rangle_{\Gamma \otimes \mathbb{R}} = \delta_{ij}. \quad (7.1)$$

By direct computation, we have

$$\langle\langle \omega_1, \omega_1 \rangle\rangle = \hat{\alpha} - \check{\alpha}^2, \quad \langle\langle \omega_2, \omega_2 \rangle\rangle = \hat{\beta} - \check{\beta}^2, \quad \langle\langle \omega_1, \omega_2 \rangle\rangle = -\check{\alpha}\check{\beta}, \quad (7.2)$$

and thus the volume of the  $\Gamma$ -Jacobian torus  $\text{Jac}^{\Gamma}$  is given by

$$\text{vol}(\text{Jac}^{\Gamma}) = \text{vol}(\text{Alb}^{\Gamma})^{-1} = \sqrt{\det(\langle\langle \omega_i, \omega_j \rangle\rangle)_{i,j=1}^2} = \sqrt{\hat{\alpha}\hat{\beta} - \hat{\alpha}\check{\beta}^2 - \check{\alpha}^2\hat{\beta}}.$$

Since  $g_0 = (\langle\sigma_i, \sigma_j\rangle_{g_0})_{i,j=1}^2$  is the inverse matrix of  $(\langle\langle \omega_i, \omega_j \rangle\rangle)_{i,j=1}^2$ , we then obtain the Albanese metric  $g_0$  on  $\Gamma \otimes \mathbb{R}$  as follows:

$$\begin{aligned} \langle\sigma_1, \sigma_1\rangle_{g_0} &= (\hat{\beta} - \check{\beta}^2)\text{vol}(\text{Alb}^{\Gamma})^2, \\ \langle\sigma_1, \sigma_2\rangle_{g_0} &= \check{\alpha}\check{\beta}\text{vol}(\text{Alb}^{\Gamma})^2, \\ \langle\sigma_2, \sigma_2\rangle_{g_0} &= (\hat{\alpha} - \check{\alpha}^2)\text{vol}(\text{Alb}^{\Gamma})^2. \end{aligned}$$

Let  $\{v_1, v_2\}$  be the orthonormal basis of  $\text{Hom}(\Gamma, \mathbb{R}) (\subset \mathcal{H}^1(X_0))$  given by the Gram-Schmidt orthonormalization of  $\{\omega_1, \omega_2\}$ . It follows from (7.2) that

$$v_1 = \frac{1}{\sqrt{\hat{\alpha} - \check{\alpha}^2}} \omega_1, \quad v_2 = \text{vol}(\text{Alb}^\Gamma) \left( \frac{\check{\alpha}\check{\beta}}{\sqrt{\hat{\alpha} - \check{\alpha}^2}} \omega_1 + \sqrt{\hat{\alpha} - \check{\alpha}^2} \omega_2 \right),$$

and thus we obtain

$$\begin{aligned} v_1(e_1) &= \frac{1}{\sqrt{\hat{\alpha} - \check{\alpha}^2}}, & v_1(e_2) &= 0, \\ v_2(e_1) &= \frac{\check{\alpha}\check{\beta} \text{vol}(\text{Alb}^\Gamma)}{\sqrt{\hat{\alpha} - \check{\alpha}^2}}, & v_2(e_2) &= \sqrt{\hat{\alpha} - \check{\alpha}^2} \text{vol}(\text{Alb}^\Gamma). \end{aligned}$$

Let  $\{\mathbf{v}_1, \mathbf{v}_2\}$  denote the dual basis of  $\{v_1, v_2\}$ . Needless to say,  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthonormal basis of  $\Gamma \otimes \mathbb{R}$ . As (7.1), we have

$$\sigma_i = \sum_{j=1}^2 \text{Hom}(\Gamma, \mathbb{R}) \langle v_j, \sigma_i \rangle_{\Gamma \otimes \mathbb{R}} \mathbf{v}_j = \sum_{j=1}^2 v_j(e_i) \mathbf{v}_j \quad (i, j = 1, 2).$$

It means that  $\sigma_i \in \Gamma \otimes \mathbb{R}$  may be identified with  $(v_1(e_i), v_2(e_i)) \in \mathbb{R}^2$ . In this sense, the standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0) \cong (\mathbb{R}^2, \{\mathbf{v}_1, \mathbf{v}_2\})$  is also given by  $\Phi_0(o(\tilde{e}_1)) = \Phi_0(o(\tilde{e}_2)) = (0, 0)$  and

$$\Phi_0(t(\tilde{e}_1)) = \frac{1}{\sqrt{\hat{\alpha} - \check{\alpha}^2}} \left( 1, \check{\alpha}\check{\beta} \text{vol}(\text{Alb}^\Gamma) \right), \quad \Phi_0(t(\tilde{e}_2)) = \left( 0, \sqrt{\hat{\alpha} - \check{\alpha}^2} \text{vol}(\text{Alb}^\Gamma) \right)$$

(see Figure 5).

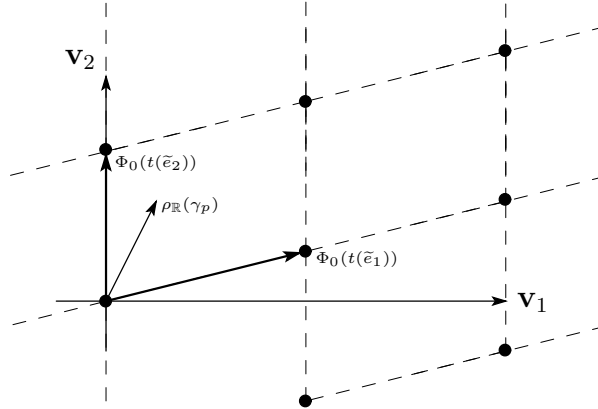


Figure 5: Modified standard realization of the square lattice graph

In particular, when the random walk is simple (i.e.,  $\alpha = \alpha' = \beta = \beta' = 1/4$ ),

$$\text{vol}(\text{Alb}^\Gamma) = 2, \quad \Phi_0(t(\tilde{e}_1)) = (\sqrt{2}, 0), \quad \Phi_0(t(\tilde{e}_2)) = (0, \sqrt{2}).$$

As mentioned before, this realization is different from the standard realization in [15, page 685] due to the difference of both the flat metric (3.3) and the invariant measure  $m$ .

## 7.2 The triangular lattice

We consider a class of non-symmetric random walks on the triangular lattice discussed in Ishiwata–Kawabi–Teruya [5]. Let  $X = (V, E)$  be a triangular lattice, where  $V = \mathbb{Z}^2$  and

$$E = \{(\mathbf{x}, \mathbf{y}) \in V \times V \mid \mathbf{y} - \mathbf{x} \in \{\pm(1, 0), \pm(0, 1), \pm(-1, 1)\}\}.$$

The transition probability of the random walk is given by

$$\begin{aligned} p((\mathbf{x}, \mathbf{x} + (1, 0))) &= \alpha, & p((\mathbf{x}, \mathbf{x} - (1, 0))) &= \alpha', \\ p((\mathbf{x}, \mathbf{x} + (0, 1))) &= \beta', & p((\mathbf{x}, \mathbf{x} - (0, 1))) &= \beta, \\ p((\mathbf{x}, \mathbf{x} + (-1, 1))) &= \gamma, & p((\mathbf{x}, \mathbf{x} - (-1, 1))) &= \gamma'. \end{aligned}$$

Here we assume

$$\alpha, \alpha', \beta, \beta', \gamma, \gamma' > 0, \quad \hat{\alpha} + \hat{\beta} + \hat{\gamma} = 1 \quad \text{and} \quad \check{\alpha} = \check{\beta} = \check{\gamma} = \kappa \geq 0.$$

The constant  $\kappa$  can be regarded the intensity of non-symmetry of the random walk, and under this assumption, the random walk on  $X$  is aperiodic (i.e.,  $K = 1$ ). We also mention that the random walk is of period  $K = 3$  when  $\alpha' = \beta' = \gamma' = 0$ . It is easy to see that  $X$  is invariant under the  $\mathbb{Z}^2$ -action generated by  $\sigma_1(\mathbf{x}) = \mathbf{x} + (1, 0)$  and  $\sigma_2(\mathbf{x}) = \mathbf{x} + (0, 1)$ . Its quotient graph  $X_0 = (V_0, E_0)$  is a 3-bouquet graph consisting of  $V_0 = \{x_0\}$ ,  $E_0 = \{e_1, e_2, e_3\}$  (see Figure 6).

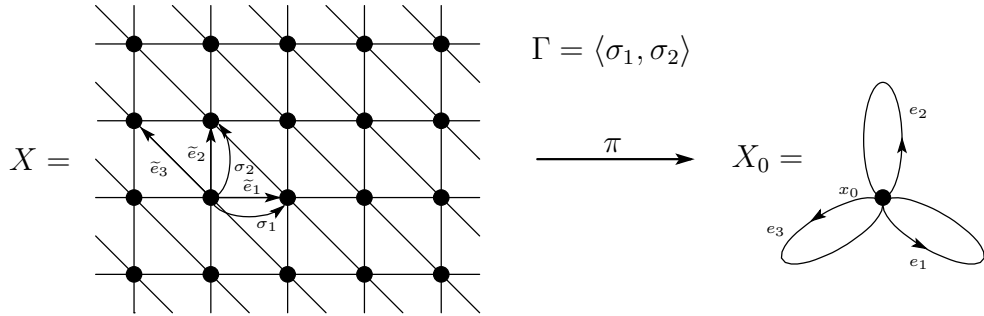


Figure 6: Triangular lattice and its quotient

The first homology group  $H_1(X_0, \mathbb{R})$  is spanned by  $[e_1], [e_2], [e_3]$ . By definition, we have

$$\gamma_p = \check{\alpha}[e_1] - \check{\beta}[e_2] + \check{\gamma}[e_3] = \kappa([e_1] - [e_2] + [e_3]).$$

Since the quotient graph  $X_0$  is a bouquet graph, we see  $H^1(X_0, \mathbb{R}) \cong \mathcal{H}^1(X_0) = C^1(X_0, \mathbb{R})$  and  $m(x_0) = 1$ . We define the canonical surjective linear map  $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R}$  by

$$\rho_{\mathbb{R}}([e_1]) := \sigma_1, \quad \rho_{\mathbb{R}}([e_2]) := \sigma_2, \quad \rho_{\mathbb{R}}([e_3]) := \sigma_2 - \sigma_1.$$

Then the asymptotic direction is given by

$$\rho_{\mathbb{R}}(\gamma_p) = (\check{\alpha} - \check{\gamma}) \sigma_1 + (\check{\gamma} - \check{\beta}) \sigma_2 = \mathbf{0},$$

and we find that  $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$  is equivalent to  $\check{\alpha} = \check{\beta} = \check{\gamma}$ .

We introduce the basis  $\{u_1, u_2\}$  in  $\text{Hom}(\Gamma, \mathbb{R})$  by

$$u_1((m, n)) = m, \quad u_2((m, n)) = n \quad \text{for } (m, n) \in \mathbb{Z}^2 \cong \Gamma.$$

Note that  $\{u_1, u_2\}$  is the dual basis of  $\{\sigma_1, \sigma_2\}$ . Let  $\{\omega_1, \omega_2, \omega_3\} (\subset H^1(X_0, \mathbb{R}))$  be the dual basis of  $\{[e_1], [e_2], [e_3]\} (\subset H_1(X_0, \mathbb{R}))$ . By direct computation, we have

$$\begin{cases} \langle\langle \omega_1, \omega_1 \rangle\rangle = \hat{\alpha} - \check{\alpha}^2 = \hat{\alpha} - \kappa^2, & \langle\langle \omega_1, \omega_2 \rangle\rangle = \check{\alpha} \check{\beta} = \kappa^2, \\ \langle\langle \omega_2, \omega_2 \rangle\rangle = \hat{\beta} - \check{\beta}^2 = \hat{\beta} - \kappa^2, & \langle\langle \omega_2, \omega_3 \rangle\rangle = \check{\beta} \check{\gamma} = \kappa^2, \\ \langle\langle \omega_3, \omega_3 \rangle\rangle = \hat{\gamma} - \check{\gamma}^2 = \hat{\gamma} - \kappa^2, & \langle\langle \omega_1, \omega_3 \rangle\rangle = -\check{\alpha} \check{\gamma} = -\kappa^2. \end{cases} \quad (7.3)$$

Since  $X$  is a non-maximal abelian covering graph of  $X_0$  with the covering transformation group  $\Gamma \cong \mathbb{Z}^2$ , we need to find a  $\mathbb{Z}$ -basis of the lattice

$$L = \{\omega \in H^1(X_0, \mathbb{Z}) \mid \omega([c]) = 0 \text{ for every cycle } \tilde{c} \text{ on } X\}.$$

It is easy to find that  $u_1 = {}^t\rho_{\mathbb{R}}(u_1) = \omega_1 - \omega_3$  and  $u_2 = {}^t\rho_{\mathbb{R}}(u_2) = \omega_2 + \omega_3$  form a  $\mathbb{Z}$ -basis of the lattice  $L$ . It follows from (7.3) that

$$\langle\langle u_1, u_1 \rangle\rangle = \hat{\alpha} + \hat{\gamma}, \quad \langle\langle u_1, u_2 \rangle\rangle = -\hat{\gamma}, \quad \langle\langle u_2, u_2 \rangle\rangle = \hat{\beta} + \hat{\gamma}, \quad (7.4)$$

and thus we obtain

$$\text{vol}(\text{Jac}^\Gamma) = \text{vol}(\text{Alb}^\Gamma)^{-1} = \sqrt{\det (\langle\langle u_i, u_j \rangle\rangle)_{i,j=1}^2} = \sqrt{\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}},$$

and the Albanese metric  $g_0$  on  $\Gamma \otimes \mathbb{R}$  as follows:

$$\begin{aligned} \langle\sigma_1, \sigma_1\rangle_{g_0} &= \frac{\hat{\beta} + \hat{\gamma}}{\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}}, \\ \langle\sigma_1, \sigma_2\rangle_{g_0} &= \frac{\hat{\gamma}}{\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}}, \\ \langle\sigma_2, \sigma_2\rangle_{g_0} &= \frac{\hat{\alpha} + \hat{\gamma}}{\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}}. \end{aligned}$$

We now determine the modified standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ . Let  $\tilde{e}_i$  ( $i = 1, 2$ ) be a lift of  $e_i$  to  $X$ , and we set  $\Phi_0(o(\tilde{e}_1)) = \Phi_0(o(\tilde{e}_2)) = \mathbf{0}$ . As in Example 7.1, we also see that

$$\Phi_0(t(\tilde{e}_1)) = \sigma_1, \quad \Phi_0(t(\tilde{e}_2)) = \sigma_2$$

is the modified harmonic realization. Let  $\{v_1, v_2\}$  be the Gram–Schmidt orthogonalization of  $\{u_1, u_2\}$ , and let  $\{\mathbf{v}_1, \mathbf{v}_2\}$  be the dual basis of  $\{v_1, v_2\}$  in  $\Gamma \otimes \mathbb{R}$ . By (7.4), we have

$$v_1 = \frac{1}{\sqrt{\hat{\alpha} + \hat{\gamma}}} u_1, \quad v_2 = \sqrt{\frac{\hat{\alpha} + \hat{\gamma}}{\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}}} \left( \frac{\hat{\gamma}}{\hat{\alpha} + \hat{\gamma}} u_1 + u_2 \right),$$

and this implies

$$\begin{aligned} v_1(e_1) &= \frac{1}{\sqrt{\hat{\alpha} + \hat{\gamma}}}, & v_1(e_2) &= 0, \\ v_2(e_1) &= \frac{\hat{\gamma} \operatorname{vol}(\operatorname{Alb}^\Gamma)}{\sqrt{\hat{\alpha} + \hat{\gamma}}}, & v_2(e_2) &= \sqrt{\hat{\alpha} + \hat{\gamma}} \operatorname{vol}(\operatorname{Alb}^\Gamma). \end{aligned}$$

Therefore the modified standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0) \cong (\mathbb{R}^2, \{\mathbf{v}_1, \mathbf{v}_2\})$  is given by  $\Phi_0(o(\tilde{e}_1)) = \Phi_0(o(\tilde{e}_2)) = (0, 0)$  and

$$\Phi_0(t(\tilde{e}_1)) = \frac{1}{\sqrt{\hat{\alpha} + \hat{\gamma}}} (1, \hat{\gamma} \operatorname{vol}(\operatorname{Alb}^\Gamma)), \quad \Phi_0(t(\tilde{e}_2)) = (0, \sqrt{\hat{\alpha} + \hat{\gamma}} \operatorname{vol}(\operatorname{Alb}^\Gamma))$$

(see Figure 7). This realization is same as in [5, page 133], and if we consider the simple random walk (i.e.,  $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = 1/6$ ),

$$\operatorname{vol}(\operatorname{Alb}^\Gamma) = \sqrt{3}, \quad \Phi_0(t(\tilde{e}_1)) = \left( \frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2} \right), \quad \Phi_0(t(\tilde{e}_2)) = (0, \sqrt{2}).$$

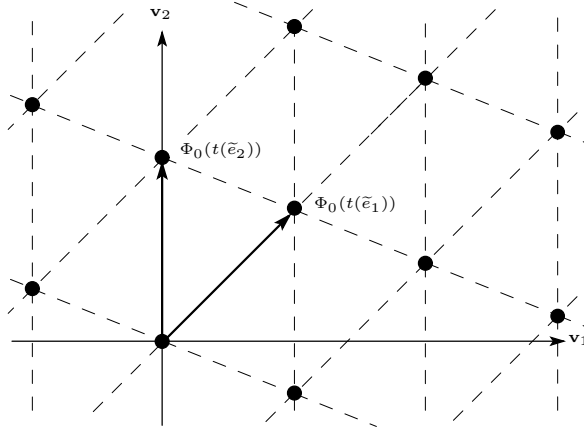


Figure 7: Modified standard realization of the triangular lattice in the case  $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$ .

Furthermore, Theorems 2.5 and 6.8 (see also [5, Theorem 2.2]) also yield the following precise asymptotics of the  $n$ -step transition probability  $p(n, x, y)$ :

$$\begin{aligned} 2\pi n \cdot p(n, x, y) &= \operatorname{vol}(\operatorname{Alb}^\Gamma) \exp \left( - \frac{|\Phi_0(y) - \Phi_0(x)|_{g_0}^2}{2n} \right) \\ &\quad \times (1 + a_1(\kappa; \Phi_0(y) - \Phi_0(x))n^{-1}) + O(n^{-3/2}) \end{aligned}$$

as  $n \rightarrow \infty$  uniformly for  $x, y \in V$  with  $|\Phi_0(y) - \Phi_0(x)|_{g_0} \leq Cn^{1/6}$ . Here the coefficient  $a_1(\kappa; \Phi_0(y) - \Phi_0(x))$  is explicitly given by

$$\begin{aligned} a_1(\kappa; \Phi_0(y) - \Phi_0(x)) = & -1 + \frac{\text{vol}(\text{Alb}^\Gamma)^4}{8} \{ \hat{\alpha}(\hat{\beta} + \hat{\gamma})^2 + \hat{\beta}(\hat{\gamma} + \hat{\alpha})^2 + \hat{\gamma}(\hat{\alpha} + \hat{\beta})^2 \} \\ & + \text{vol}(\text{Alb}^\Gamma)^4 \left\{ (\hat{\alpha}\hat{\beta} - 2\hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha})(\Phi_0(y) - \Phi_0(x))_1 \right. \\ & \quad \left. + (-\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\gamma} + 2\hat{\gamma}\hat{\alpha})(\Phi_0(y) - \Phi_0(x))_2 \right\} \kappa \\ & + \frac{3}{8} \text{vol}(\text{Alb}^\Gamma)^4 \left( -1 + 5\hat{\alpha}\hat{\beta}\hat{\gamma} \text{vol}(\text{Alb}^\Gamma)^2 \right) \kappa^2. \end{aligned}$$

### 7.3 The hexagonal lattice

We discuss the modified standard realization of the hexagonal lattice  $X = (V, E)$ , where  $V = \mathbb{Z}^2 = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}\}$  and

$$E = \{(\mathbf{x}, \mathbf{y}) \in V \times V \mid \mathbf{x} - \mathbf{y} = \pm(1, 0), \mathbf{x} - \mathbf{y} = (0, (-1)^{x_1+x_2})\}$$

(see Figure 8).

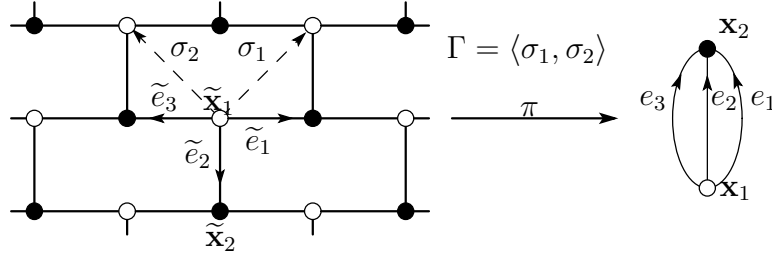


Figure 8: Hexagonal lattice and its quotient

We consider a random walk on  $X$  given in the following manner: If  $\mathbf{x} = (x_1, x_2) \in V$  is a vertex so that  $x_1 + x_2$  is even, then we set

$$p(\mathbf{x}, \mathbf{x} + (1, 0)) = \alpha, \quad p(\mathbf{x}, \mathbf{x} - (0, 1)) = \beta, \quad p(\mathbf{x}, \mathbf{x} - (1, 0)) = \gamma$$

with  $\alpha, \beta, \gamma \geq 0$  and  $\alpha + \beta + \gamma = 1$ . If  $x_1 + x_2$  is odd, then we set

$$p(\mathbf{x}, \mathbf{x} - (1, 0)) = \alpha', \quad p(\mathbf{x}, \mathbf{x} + (0, 1)) = \beta', \quad p(\mathbf{x}, \mathbf{x} + (1, 0)) = \gamma'$$

with  $\alpha', \beta', \gamma' \geq 0$  and  $\alpha' + \beta' + \gamma' = 1$ . Here we see that  $X$  is invariant under the action  $\Gamma = \langle \sigma_1, \sigma_2 \rangle (\cong \mathbb{Z}^2)$  generated by  $\sigma_1(\mathbf{x}) = \mathbf{x} + (1, 1)$  and  $\sigma_2(\mathbf{x}) = \mathbf{x} + (-1, 1)$ . Then the quotient of  $X$  by the action  $\Gamma$  is a finite graph  $X_0 = (V_0, E_0)$  consisting of two vertices  $V_0 = \{x_1, x_2\}$  with three multiple edges  $E = E_{x_1} \cup E_{x_2} = \{e_1, e_2, e_3\} \cup \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ . The transition probability of the corresponding random walk on  $X_0$  is given by

$$p(e_1) = \alpha, \quad p(\bar{e}_1) = \alpha', \quad p(e_2) = \beta, \quad p(\bar{e}_2) = \beta', \quad p(e_3) = \gamma, \quad p(\bar{e}_3) = \gamma'.$$

The first homology group  $H_1(X_0, \mathbb{R})$  is spanned by the two cycles  $[c_1] := [e_1 * \bar{e}_2]$  and  $[c_2] := [e_3 * \bar{e}_2]$ . Solving (2.1), we obtain  $m(x_1) = m(x_2) = 1/2$ . Thus the homological direction  $\gamma_p$  is given by

$$\gamma_p = \frac{1}{2} \{(\alpha - \alpha')e_1 + (\beta - \beta')e_2 + (\gamma - \gamma')e_3\} = \frac{\check{\alpha}}{2}[c_1] + \frac{\check{\gamma}}{2}[c_2].$$

We define the canonical surjective linear map  $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R}$  by

$$\rho_{\mathbb{R}}([c_1]) := \sigma_1, \quad \rho_{\mathbb{R}}([c_2]) := \sigma_2.$$

Then the asymptotic direction is given by

$$\rho_{\mathbb{R}}(\gamma_p) = \frac{\check{\alpha}}{2}\sigma_1 + \frac{\check{\gamma}}{2}\sigma_2. \quad (7.5)$$

We are going to determine the modified standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ . Set  $\tilde{\mathbf{x}}_1 = (0, 0)$  and  $\tilde{\mathbf{x}}_2 = (0, 1)$  in  $V$ . Without loss of generality, we may put  $\Phi_0(\tilde{\mathbf{x}}_1) = \mathbf{0} \in \Gamma \otimes \mathbb{R}$ . It follows from equation (2.2) that

$$\Phi_0(\tilde{\mathbf{x}}_1) - \alpha'\sigma_1 - \gamma'\sigma_2 = \Phi_0(\tilde{\mathbf{x}}_2) + \rho_{\mathbb{R}}(\gamma_p). \quad (7.6)$$

Combining (7.5) with (7.6), we obtain

$$\Phi_0(\tilde{\mathbf{x}}_1) = \mathbf{0}, \quad \Phi_0(\tilde{\mathbf{x}}_2) = -\frac{\hat{\alpha}}{2}\sigma_1 - \frac{\hat{\gamma}}{2}\sigma_2. \quad (7.7)$$

We define the basis  $\{\omega_1, \omega_2\}$  in  $\text{Hom}(\Gamma, \mathbb{R})$  by

$$\omega_1((m - n, m + n)) = m, \quad \omega_2((m - n, m + n)) = n \quad \text{for } (m, n) \in \mathbb{Z}^2 \cong \Gamma.$$

It is the dual basis of  $\{\sigma_1, \sigma_2\}$ . Identifying  $\omega_i \in \text{Hom}(\Gamma, \mathbb{R})$  ( $i = 1, 2$ ) with  ${}^t\rho_{\mathbb{R}}(\omega_i) \in H^1(X_0, \mathbb{R})$  as before, we can also see that  $\{\omega_1, \omega_2\}$  is the dual basis  $\{[c_1], [c_2]\} \subset H_1(X_0, \mathbb{R})$ . Recalling  $H^1(X_0, \mathbb{R}) \cong \mathcal{H}^1(X_0) \subset C^1(X_0, \mathbb{R})$ , we have

$$\begin{aligned} \omega_1(e_1) - \omega_1(e_2) &= 1, & \omega_1(e_3) - \omega_1(e_2) &= 0, \\ \omega_2(e_1) - \omega_2(e_2) &= 0, & \omega_2(e_3) - \omega_2(e_2) &= 1. \end{aligned} \quad (7.8)$$

By definition of the modified harmonicity (3.2), we observe that  $\omega_1$  and  $\omega_2$  also satisfy

$$\hat{\alpha}\omega_1(e_1) + \hat{\beta}\omega_1(e_2) + \hat{\gamma}\omega_1(e_3) = 0 \quad \text{and} \quad \hat{\alpha}\omega_2(e_1) + \hat{\beta}\omega_2(e_2) + \hat{\gamma}\omega_2(e_3) = 0, \quad (7.9)$$

respectively. Solving the algebraic equations (7.8) and (7.9), we obtain

$$\begin{aligned} \omega_1(e_1) &= 1 - \frac{\hat{\alpha}}{2} = \frac{\hat{\beta} + \hat{\gamma}}{2}, & \omega_1(e_2) &= \omega_1(e_3) = -\frac{\hat{\alpha}}{2}, \\ \omega_2(e_1) &= \omega_2(e_2) = -\frac{\hat{\gamma}}{2}, & \omega_2(e_3) &= 1 - \frac{\hat{\gamma}}{2} = \frac{\hat{\alpha} + \hat{\beta}}{2}. \end{aligned}$$

Then by direct computation, we have

$$\langle\langle\omega_1, \omega_1\rangle\rangle = \frac{\hat{\alpha}(\hat{\beta} + \hat{\gamma}) - \check{\alpha}^2}{4}, \quad \langle\langle\omega_1, \omega_2\rangle\rangle = -\frac{\hat{\alpha}\hat{\gamma} + \check{\alpha}\check{\gamma}}{4}, \quad \langle\langle\omega_2, \omega_2\rangle\rangle = \frac{(\hat{\alpha} + \hat{\beta})\hat{\gamma} - \check{\gamma}^2}{4}.$$

Thus the volume of the  $\Gamma$ -Jacobian torus  $\text{Jac}^\Gamma$  and the Albanese metric  $g_0$  on  $\Gamma \otimes \mathbb{R}$  are given by

$$\begin{aligned} \text{vol}(\text{Jac}^\Gamma) &= \text{vol}(\text{Alb}^\Gamma)^{-1} = \sqrt{\det(\langle\langle\omega_i, \omega_j\rangle\rangle)_{i,j=1}^2} \\ &= \frac{\sqrt{2\hat{\alpha}\hat{\beta}\hat{\gamma} + \hat{\alpha}\check{\alpha}(\hat{\beta}\check{\gamma} + \check{\beta}\hat{\gamma}) + \check{\beta}\check{\beta}(\hat{\alpha}\check{\gamma} + \check{\alpha}\hat{\gamma}) + \hat{\gamma}\check{\gamma}(\hat{\alpha}\check{\beta} + \check{\alpha}\hat{\beta})}}{4}, \end{aligned}$$

and

$$\begin{aligned} \langle\sigma_1, \sigma_1\rangle_{g_0} &= \frac{\text{vol}(\text{Alb}^\Gamma)^2}{4} \{(\hat{\alpha} + \hat{\beta})\hat{\gamma} - \check{\gamma}^2\}, \\ \langle\sigma_1, \sigma_2\rangle_{g_0} &= \frac{\text{vol}(\text{Alb}^\Gamma)^2}{4} (\hat{\alpha}\hat{\gamma} + \check{\alpha}\check{\gamma}), \\ \langle\sigma_2, \sigma_2\rangle_{g_0} &= \frac{\text{vol}(\text{Alb}^\Gamma)^2}{4} \{\hat{\alpha}(\hat{\beta} + \hat{\gamma}) - \check{\alpha}^2\}, \end{aligned}$$

respectively.

Let  $\{v_1, v_2\}$  be the Gram–Schmidt orthogonalization of  $\{\omega_1, \omega_2\}$ , that is,

$$v_1 = \frac{\omega_1}{\|\omega_1\|}, \quad v_2 = \text{vol}(\text{Alb}^\Gamma) \|\omega_1\| \left( \omega_2 - \frac{\langle\langle\omega_2, \omega_1\rangle\rangle}{\|\omega_1\|^2} \omega_1 \right).$$

Recalling that  $\{\omega_1, \omega_2\}$  is the dual basis of  $\{[c_1], [c_2]\}$ , we obtain

$$v_1([c_i]) = \frac{\delta_{i1}}{\|\omega_1\|}, \quad v_2([c_i]) = \text{vol}(\text{Alb}^\Gamma) \|\omega_1\| \left( \delta_{i2} - \frac{\langle\langle\omega_2, \omega_1\rangle\rangle}{\|\omega_1\|^2} \delta_{i1} \right) \quad (i = 1, 2),$$

and hence

$$\begin{aligned} \sigma_1 &= v_1([c_1]) \mathbf{v}_1 + v_2([c_1]) \mathbf{v}_2 = \frac{1}{\|\omega_1\|} \mathbf{v}_1 - \frac{\langle\langle\omega_1, \omega_2\rangle\rangle \text{vol}(\text{Alb}^\Gamma)}{\|\omega_1\|} \mathbf{v}_2, \\ \sigma_2 &= v_2([c_1]) \mathbf{v}_1 + v_2([c_2]) \mathbf{v}_2 = \|\omega_1\| \text{vol}(\text{Alb}^\Gamma) \mathbf{v}_2. \end{aligned} \tag{7.10}$$

Combining (7.7) with (7.10), we finally find that the modified standard realization  $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0) \cong (\mathbb{R}^2, \{\mathbf{v}_1, \mathbf{v}_2\})$  is given by  $\Phi_0(\tilde{\mathbf{x}}_1) = (0, 0)$  and

$$\Phi_0(\tilde{\mathbf{x}}_2) = \left( -\frac{\hat{\alpha}}{\sqrt{\hat{\alpha}(\hat{\beta} + \hat{\gamma}) - \check{\alpha}^2}}, \frac{-2\hat{\alpha}\hat{\gamma} - \hat{\alpha}\check{\alpha}\check{\gamma} + \check{\alpha}^2\hat{\alpha}}{4\sqrt{\hat{\alpha}(\hat{\beta} + \hat{\gamma}) - \check{\alpha}^2}} \text{vol}(\text{Alb}^\Gamma) \right)$$

(see Figure 9). In particular, when the random walk is simple (i.e.,  $\alpha = \alpha' = \beta = \beta' =$



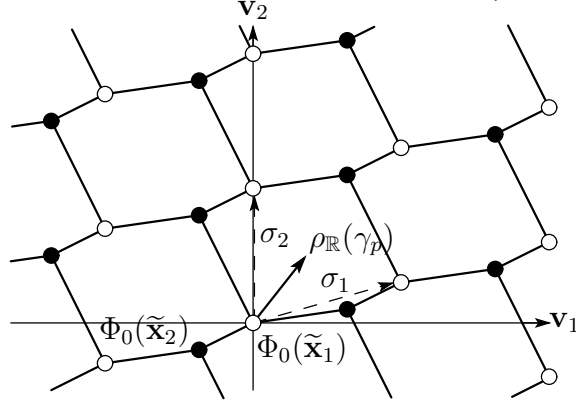


Figure 9: Modified standard realization of the hexagonal lattice

$\gamma = \gamma' = 1/3$ ), we have

$$\text{vol}(\text{Alb}^\Gamma) = 3\sqrt{3}, \quad \Phi_0(\tilde{\mathbf{x}}_2) = \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2} \right), \quad \sigma_1 = \left( \frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2} \right), \quad \sigma_2 = (0, \sqrt{6}).$$

It means that the shape of the fundamental pattern of  $\Phi_0(X)$  is the equilateral hexagonal lattice with the common length of the sides  $\sqrt{2}$ .

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